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# Aharonov–Bohm quantum systems on a punctured 2-torus

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**Abstract**
This paper deals with Aharonov–Bohm (A-B) quantum systems on a punctured two-dimensional torus from the geometric and the operator theoretic point of view. First, flat connections on the $U(1)$-bundles over the punctured 2-torus are studied, which serve as vector potentials for A-B effect magnetic fields. It is proved that the moduli space of flat connections is identified with the $(N+1)$-dimensional torus $T^{N+1}$, if the punctured torus has $N > 0$ pinholes at which solenoids are assumed to penetrate the 2-torus. For a given point of $T^{N+1}$, an associated flat connection is constructed in terms of the Weierstrass zeta function on $\mathbf{C}$ together with an inhomogeneous linear function on $\mathbf{R}^2$. A-B quantum systems are defined in terms of position operators and momentum operators coupled with the A-B potentials. Necessary and sufficient conditions are given for two A-B quantum systems to be unitarily equivalent. Further, the A-B Hamiltonian is defined and analysed from the viewpoint of operator theory. The deficiency indices of the A-B Hamiltonian are determined to be $(N + M, N + M)$, where $M$ is the number of solenoids whose fluxes are not quantized. Finally, the eigenvalue problem is studied for the A-B Hamiltonian with all fluxes quantized to obtain eigenvalues together with eigenfunctions which are described in terms of the Weierstrass sigma functions.

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## 1. Introduction

The Aharonov–Bohm effect (A-B effect) is known as a topological effect that gives rise to an observable phase shift in the wavefunction. This effect was predicted by Aharonov and Bohm [3] and verified finally by Tonomura *et al* [25] through an elaborate and precise experiment.

The physical setting for the A-B effect is that a charged particle moves outside the solenoid sitting along the $x_3$-axis, where the radius of the solenoid is assumed to tend to zero, while the total flux of the solenoid is kept constant. Then, the vector potential $\boldsymbol{A}$ associated with the solenoid is singular on the $x_3$-axis and satisfies $\boldsymbol{B} = \mathrm{rot}\,\boldsymbol{A} = 0$ on $\mathbf{R}^3\backslash\{x_3\text{-axis}\}$. However, the wavefunction for the charged particle gets influenced in its phase by the factor

$$\exp\left(\mathrm{i}\int_\ell \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{x}\right), \tag{1.0.1}$$

where $\ell$ is a loop which goes around the solenoid. This quantum system is reduced to a two-dimensional one because of the translation symmetry in the direction of the $x_3$-axis. The vector potential is then interpreted in differential geometry as a connection form on the product bundle $(\mathbf{R}^2\backslash\{\mathbf{0}\}) \times U(1)$, and the phase factor as the holonomy along the loop $\ell$ with respect to $\boldsymbol{A}$. (See [2, 15].)

Since momentum operators are coupled with the vector potential, the A-B vector potential makes canonical commutation relations changed [17], but the change does not appear explicit in the canonical commutation relations because of $\mathrm{rot}\,\boldsymbol{A} = 0$. It can be well observed when commutators among the unitary operators generated by the momentum operators are taken into account. In a series of papers [5] on the A-B effect in the case of $N$ solenoids, Arai studied actually unitary operators generated by momentum operators coupled with a vector potential singular at $N$ distinct points in $\mathbf{R}^2$. He also gave an example of a vector potential singular at points in an infinite lattice by using the Weierstrass zeta function [6].

As for the canonical commutation relation among momentum operators coupled with the vector potential, the magnetic translation group is of interest [11]. Tanimura *et al* [22–24] also studied the magnetic translation group in the $n$-dimensional torus.

The Aharonov–Bohm Hamiltonian (A-B Hamiltonian) has been discussed by a number of researchers. Adami and Teta [1] analysed the A-B Hamiltonian from the viewpoint of operator theory to describe the spectrum, the generalized eigenfunctions and the scattering amplitude. Nambu [16] considered the $N$-solenoid A-B Hamiltonian perturbed by a uniform magnetic field. He constructed the explicit eigenfunctions and investigated their properties in the limit that the perturbing uniform magnetic field tends to zero. Mine [14] also analysed the same Hamiltonian as Nambu studied, to obtain the deficiency indices of the Hamiltonian and the distribution of the eigenvalues by means of the so-called localization principle, where the localization principle is intuitively explained as follows: each singularity makes a separate contribution to the total deficiency index of the operator in question. Papers on related subjects other than that mentioned above will be referred to in the text and in conclusion of this paper.

The interest of this paper rises from stimulating papers [6, 22]. The A-B potential in [6] and the circle bundles over the 2-torus in [22] will be put together to set up quantum systems exhibiting the A-B effect on a punctured 2-torus, where the punctured 2-torus means a torus with a finite number of pinholes at which the 2-torus is assumed to be penetrated by solenoids. The A-B potential is then generalized to flat connections on the punctured 2-torus. Further, the moduli space of flat connections is identified with the $(N+1)$-torus $T^{N+1}$, where $N$ is the number of singularities of the flat connection, or the number of the solenoids. On the basis of the connection theory, quantum mechanics on the punctured 2-torus will be set up. The A-B quantum system is assigned by position and momentum operators, where the momentum operator is assumed to be coupled with the A-B potential. Necessary and sufficient conditions will be given for two quantum systems on the punctured 2-torus to be unitarily equivalent. The coupled momentum operators build up the A-B Hamiltonian. The deficiency indices of the A-B Hamiltonian are determined in terms of fluxes of the solenoids by means of the localization principle together with the well-known fact on the deficiency indices of the usual A-B Hamiltonian. It turns out that the deficiency indices of the A-B Hamiltonian in question are $(N+M, N+M)$, where $M$ is the number of non-quantized fluxes, where a flux $\nu$ is called quantized if $n\nu$ is an integer, $n$ being the number associated with the complex line bundle over the punctured 2-torus. In addition, eigenvalues and the associated eigenfunctions are obtained for the adjoint operator to the A-B Hamiltonian.

This paper is broken up into two main parts, the first of which, section 2, is concerned with flat connections on the $U(1)$-bundles over a punctured 2-torus, and the other, section 3, with quantum systems on the punctured 2-torus. Section 2.1 contains a brief review of the $U(1)$-bundles over the punctured 2-torus. The bundle equivalence of $U(1)$-bundles is also discussed. Section 2.2 deals with the connections on the $U(1)$-bundles over the punctured 2-torus. In particular, three examples of flat connections are given. One of them is a gauge potential for a uniform magnetic field, and the others are connections constructed from the Weierstrass zeta function and from the Weierstrass $\wp$-function. In section 2.3, it is shown that the sum of fluxes at singular points of a flat connection is equal to the integer that characterizes the $U(1)$-bundle on which the flat connection is defined. In section 2.4, the gauge group on the $U(1)$-bundles is studied and identified; the gauge group is shown to be isomorphic with the group of $U(1)$-valued functions with periodicity. In section 2.5, by making full use of the gauge group, necessary and sufficient conditions are given for two flat connections to be gauge equivalent. Some of conditions are described in terms of fluxes of solenoids, and the others are concerned with the cycles of the 2-torus. Section 2.6 contains the moduli space of flat connections, which is identified with $T^{N+1}$ if the punctured 2-torus has $N > 0$ pinholes. If there is no pinhole, the moduli space of flat connections is identified with $T^2$. The proof of sufficiency is carried out by constructing a flat connection associated with a point of $T^{N+1}$ by means of the Weierstrass zeta function together with an inhomogeneous linear function on $\mathbf{R}^2$. If $N = 0$, an associated flat connection is formed from an inhomogeneous linear function on $\mathbf{R}^2$. Section 2.7 contains a remark on the holonomy of the flat connection.

In section 3, quantum mechanics on the punctured 2-torus is dealt with. In section 3.1, complex line bundles associated with the $U(1)$-bundles are defined. In section 3.2, sections in the complex line bundles are studied. As is well known, the sections in the complex line bundles are in one-to-one correspondence with the equivariant functions on the $U(1)$-bundles. Further, the equivariant functions are identified with functions satisfying a shift condition on the punctured plane. Operators acting on sections in the complex line bundles over the punctured 2-torus will be put into those acting on functions satisfying the shift condition on the punctured plane. Section 3.3 deals with covariant differential operators. The covariant derivative of a section in the complex line bundle is associated with the 'covariant' derivative of the corresponding function satisfying the shift condition. Section 3.4 contains Hilbert spaces which are unitarily isomorphic to one another, each of which will serve as the space of wavefunctions on the punctured 2-torus. In section 3.5, a quantum system on the punctured 2-torus is defined by assigning position operators and momentum operators coupled with the A-B potential. The commutator of the unitary operators generated by the coupled momentum operators is given in terms of fluxes of solenoids. In section 3.6, necessary and sufficient conditions are given for two A-B quantum systems on the punctured 2-torus to be unitarily equivalent. These conditions are a representation of those conditions for two flat connections to be gauge equivalent. In section 3.7, holonomy in the A-B quantum system is touched upon. In section 3.8, it is shown by means of the localization principle that the deficiency indices of the A-B Hamiltonian on the punctured 2-torus are $(N + M, N + M)$, where $M$ is the number of non-quantized fluxes. For the adjoint operator to the A-B Hamiltonian with all fluxes quantized, the eigenvalue problem is studied, in section 3.9, to obtain eigenvalues together with the associated eigenfunctions described in terms of the Weierstrass sigma function. Section 3.10 contains summary and concluding remarks. In the appendix, the proof of the localization lemma is given.

## 2. Flat connections

After defining $U(1)$-bundle over a punctured 2-torus, we discuss flat connections on the $U(1)$-bundle along with the gauge group. It will be shown that the moduli space of the flat connections is identified with $T^{N+1}$, where $N$ is the number of pinholes of the punctured 2-torus. The flat connection is interpreted as describing an A-B potential with singularity at pinholes.

### 2.1. $U(1)$-bundles over a punctured 2-torus

Let $\boldsymbol{c}_{\boldsymbol{0}}^{(j)}$, $j = 1, \ldots, N$, be $N$ distinct points in the square $[0, 1) \times [0, 1)$, where $N$ is a non-negative integer. We translate parallel these points by $\boldsymbol{m} \in \mathbf{Z}^2$ on the plane $\mathbf{R}^2$ and set

$$\boldsymbol{c}_{\boldsymbol{m}}^{(j)} = \boldsymbol{c}_{\boldsymbol{0}}^{(j)} + \boldsymbol{m}, \qquad j = 1, \ldots, N, \qquad \boldsymbol{m} \in \mathbf{Z}^2. \tag{2.1.1}$$

Thus, we have the lattice, $\Lambda$, of points $\boldsymbol{c}_{\boldsymbol{m}}^{(j)}$ and the punctured plane

$$\dot{\mathbf{R}}^2 = \mathbf{R}^2 \backslash \Lambda, \qquad \Lambda = \left\{ \boldsymbol{c}_{\boldsymbol{m}}^{(j)} \middle| j = 1, \ldots, N, \boldsymbol{m} \in \mathbf{Z}^2 \right\}. \tag{2.1.2}$$

If $N = 0$, we think of $\Lambda$ as empty. The equivalence relation defined on $\dot{\mathbf{R}}^2$ through $\boldsymbol{x} \sim \boldsymbol{y} \Leftrightarrow \boldsymbol{x} - \boldsymbol{y} \in \mathbf{Z}^2$ determines a punctured 2-torus, which we denote by $\dot{T}^2$,

$$p : \dot{\mathbf{R}}^2 \longrightarrow \dot{T}^2 = \dot{\mathbf{R}}^2 / \mathbf{Z}^2, \tag{2.1.3}$$

where $p$ is the natural projection.

Let $M(2, \mathbf{Z})$ denote the set of $2 \times 2$ matrices with integer entries. For a given $\omega \in M(2, \mathbf{Z})$, we define a multiplication operation on $\mathbf{R} \times \mathbf{R}^2$ by

$$(x_0, \boldsymbol{x}) \cdot (y_0, \boldsymbol{y}) = (x_0 + y_0 + \langle \boldsymbol{x}, \omega \boldsymbol{y} \rangle, \boldsymbol{x} + \boldsymbol{y}), \qquad (x_0, \boldsymbol{x}), (y_0, \boldsymbol{y}) \in \mathbf{R} \times \mathbf{R}^2, \tag{2.1.4}$$

where $\langle \cdot , \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^2$. Then the product space $\mathbf{R} \times \mathbf{R}^2$ becomes a group, of which the identity is given by $(0, \mathbf{0})$ and the inverse of $(x_0, \boldsymbol{x})$ by $(-x_0 + \langle \boldsymbol{x}, \omega \boldsymbol{x} \rangle, -\boldsymbol{x})$. We denote this group by $\mathbf{R} \times {}_{\omega}\mathbf{R}^2$, which is a central extension of $\mathbf{R}^2$.

The subgroup $\mathbf{Z} \times {}_{\omega}\mathbf{Z}^2$ of $\mathbf{R} \times {}_{\omega}\mathbf{R}^2$ acts on $\mathbf{R} \times \dot{\mathbf{R}}^2$ to the left; for $(m_0, \boldsymbol{m}) \in \mathbf{Z} \times {}_{\omega}\mathbf{Z}^2$, we denote the left action of $(m_0, \boldsymbol{m})$ by $L_{(m_0, \boldsymbol{m})}$,

$$L_{(m_0, \boldsymbol{m})}(x_0, \boldsymbol{x}) = (x_0 + m_0 + \langle \boldsymbol{m}, \omega \boldsymbol{x} \rangle, \boldsymbol{x} + \boldsymbol{m}), \qquad (x_0, \boldsymbol{x}) \in \mathbf{R} \times \dot{\mathbf{R}}^2. \tag{2.1.5}$$

Since this action is free, the factor space becomes a manifold, which we denote by $\dot{P}^3_{\omega}$,

$$\Pi_{\omega} : \mathbf{R} \times \dot{\mathbf{R}}^2 \longrightarrow \dot{P}^3_{\omega} = (\mathbf{Z} \times {}_{\omega}\mathbf{Z}^2) \backslash (\mathbf{R} \times \dot{\mathbf{R}}^2), \tag{2.1.6}$$

where $\Pi_{\omega}$ is the natural projection. We denote by $[(x_0, \boldsymbol{x})]$ the equivalence class with a representative $(x_0, \boldsymbol{x}) \in \mathbf{R} \times \dot{\mathbf{R}}^2$, so that we have $\Pi_{\omega}(x_0, \boldsymbol{x}) = [(x_0, \boldsymbol{x})]$. The manifold $\dot{P}^3_{\omega}$ admits also the right action of $U(1)$,

$$R_g : [(x_0, \boldsymbol{x})] \longmapsto [(x_0, \boldsymbol{x})] \cdot \mathrm{e}^{2\pi \mathrm{i} t} = [(x_0 + t, \boldsymbol{x})], \qquad g = \mathrm{e}^{2\pi \mathrm{i} t} \in U(1). \tag{2.1.7}$$

This action is free, as is easily seen. Thus, the manifold $\dot{P}^3_{\omega}$ is made into a $U(1)$-bundle over $\dot{T}^2 \cong \dot{P}^3_{\omega} / U(1)$,

$$\pi_{\omega} : \dot{P}^3_{\omega} \longrightarrow \dot{T}^2 \cong \dot{P}^3_{\omega} / U(1), \tag{2.1.8}$$

where $\pi_\omega$ is the natural projection. If $N = 0$, the bundle $\dot{P}^3_\omega$ is the same as Tanimura [22] defined and called a magnetic fibre bundle, $U(1) \to P^3_\omega \to T^2$, where the dot is deleted, as $\Lambda$ is empty. The following diagram makes the construction of this bundle comprehensible:

$$\begin{array}{ccccc} \mathbf{Z} \times_\omega \{\mathbf{0}\} & \longrightarrow & \mathbf{Z} \times_\omega \mathbf{Z}^2 & \longrightarrow & \mathbf{Z}^2 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbf{R} \times_\omega \{\mathbf{0}\} & \longrightarrow & \mathbf{R} \times \dot{\mathbf{R}}^2 & \longrightarrow & \dot{\mathbf{R}}^2, \\ \big\downarrow & & {\scriptstyle \Pi_\omega}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ \mathrm{U}(1) & \longrightarrow & \dot{P}^3_\omega & \xrightarrow{\pi_\omega} & \dot{T}^2 \end{array} \tag{2.1.9}$$

where the sequences sitting in the top and middle rows of this diagram mean that $\mathbf{Z}^2 \cong (\mathbf{Z} \times {}_\omega\{\mathbf{0}\}) \backslash (\mathbf{Z} \times {}_\omega \mathbf{Z}^2)$ and $\dot{\mathbf{R}}^2 \cong (\mathbf{R} \times {}_\omega\{\mathbf{0}\}) \backslash (\mathbf{R} \times \dot{\mathbf{R}}^2)$, respectively.

We note here that

$$\Pi_\omega \circ L_{(m_0, \boldsymbol{m})} = \Pi_\omega \tag{2.1.10}$$

and

$$R_g \circ \Pi_\omega = \Pi_\omega \circ L_{(t,\mathbf{0})}, \qquad g = \mathrm{e}^{2\pi \mathrm{i} t}, \tag{2.1.11}$$

where $L_{(t,\mathbf{0})}$ denotes the action of $(t, \mathbf{0}) \in \mathbf{R} \times {}_\omega\{\mathbf{0}\}$ on $\mathbf{R} \times \dot{\mathbf{R}}^2$,

$$L_{(t,\mathbf{0})} : (x_0, \boldsymbol{x}) \longmapsto (t, \mathbf{0}) \cdot (x_0, \boldsymbol{x}) = (x_0 + t, \boldsymbol{x}). \tag{2.1.12}$$

We remark that it does not matter to which side the group $\mathbf{R} \times {}_\omega\{\mathbf{0}\}$ acts on $\mathbf{R} \times \dot{\mathbf{R}}^2$ and that no confusion will be caused if we use the same symbol $L_{(\cdot,\cdot)}$ to denote the left actions of $\mathbf{Z} \times {}_\omega \mathbf{Z}^2$ and of $\mathbf{R} \times {}_\omega\{\mathbf{0}\}$.

If the groups $\mathbf{Z} \times {}_\omega \mathbf{Z}^2$ and $\mathbf{Z} \times {}_{\omega'} \mathbf{Z}^2$ are isomorphic to each other under an isomorphism of the form $(m_0, \boldsymbol{m}) \mapsto (m_0 + \psi(\boldsymbol{m}), \boldsymbol{m})$ with $\psi(\boldsymbol{m})$ integer-valued, then $\omega - \omega'$ is symmetric. Conversely, if $\omega - \omega'$ is symmetric, one can find an isomorphism of $\mathbf{Z} \times {}_\omega \mathbf{Z}^2$ to $\mathbf{Z} \times {}_{\omega'} \mathbf{Z}^2$. The group isomorphism gives rise to a bundle isomorphism of $\dot{P}^3_\omega$ and $\dot{P}^3_{\omega'}$. It then turns out that $\dot{P}^3_\omega$ and $\dot{P}^3_{\omega'}$ are isomorphic, if and only if $\omega - \omega'$ is symmetric [22]. Thus, the set of inequivalent $U(1)$-bundles $\dot{P}^3_\omega$ is in one-to-one correspondence with $M(2, \mathbf{Z})/\mathrm{Sym}(2, \mathbf{Z})$, where $\mathrm{Sym}(2, \mathbf{Z})$ denotes the set of $2 \times 2$ symmetric matrices with integer entries. In other words, the bundle $\dot{P}^3_\omega$ is characterized by the anti-symmetric part of $\omega$, and hence by $\omega_{21} - \omega_{12}$, in particular. In fact, the matrix $\omega$ is broken up into

$$\omega = \begin{pmatrix} 0 & \omega_{12} - \omega_{21} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \omega_{21} \\ \omega_{21} & 0 \end{pmatrix} + \begin{pmatrix} \omega_{11} & 0 \\ 0 & \omega_{22} \end{pmatrix}, \tag{2.1.13}$$

and for the symmetric matrices on the right-hand side, the respective group isomorphisms, $(m_0, \boldsymbol{m}) \mapsto (m_0 + \psi(\boldsymbol{m}), \boldsymbol{m})$, are determined by

$$\psi_\tau(\boldsymbol{m}) = \tfrac{1}{2}\langle \boldsymbol{m}, \tau \boldsymbol{m} \rangle, \qquad \psi_\Delta(\boldsymbol{m}) = \tfrac{1}{2}(\langle \boldsymbol{m}, \Delta \boldsymbol{m} \rangle + \langle \boldsymbol{\delta}, \boldsymbol{m} \rangle), \tag{2.1.14}$$

where

$$\tau = \begin{pmatrix} 0 & \omega_{21} \\ \omega_{21} & 0 \end{pmatrix}, \qquad \Delta = \begin{pmatrix} \omega_{11} & 0 \\ 0 & \omega_{22} \end{pmatrix}, \qquad \boldsymbol{\delta} = \begin{pmatrix} \omega_{11} \\ \omega_{22} \end{pmatrix}. \tag{2.1.15}$$

The group isomorphisms determined by $\psi_\tau$ and $\psi_\Delta$ are put together to provide a bundle isomorphism of $\dot{P}^3_{\omega'}$ to $\dot{P}^3_\omega$ with $\omega' = \left(\begin{smallmatrix} 0 & \omega_{12} - \omega_{21} \\ 0 & 0 \end{smallmatrix}\right)$.

**Proposition 2.1.** *The $U(1)$-bundle $\dot{P}^3_\omega$ is characterized by the number $\omega_{21} - \omega_{12}$.*

We take the number $\omega_{21} - \omega_{12}$ in place of $\omega_{12} - \omega_{21}$, since $\omega_{21} - \omega_{12}$ will be interpreted as the sum of fluxes (see proposition 2.5).

*2.2. Connections*

For $\boldsymbol{m} \in \mathbf{Z}^2$, we denote by $T_{\boldsymbol{m}}$ the translation on $\dot{\mathbf{R}}^2$,

$$T_{\boldsymbol{m}} : \dot{\mathbf{R}}^2 \longrightarrow \dot{\mathbf{R}}^2; \qquad \boldsymbol{x} \longmapsto \boldsymbol{x} + \boldsymbol{m}. \tag{2.2.1}$$

We define a set of 1-form on $\dot{\mathbf{R}}^2$ to be

$$\mathcal{A}_\omega(\dot{\mathbf{R}}^2) = \{A : \text{1-form on } \dot{\mathbf{R}}^2 \mid T_{\boldsymbol{m}}^* A = A - \langle \boldsymbol{m}, \omega\, \mathrm{d}\boldsymbol{x}\rangle, \boldsymbol{m} \in \mathbf{Z}^2\}. \tag{2.2.2}$$

We will refer to the property satisfied by $A$ as the shift property. For $A \in \mathcal{A}_\omega(\dot{\mathbf{R}}^2)$, we define a $\mathfrak{u}(1)$-valued 1-form $\tilde{\alpha}_A$ on $\mathbf{R} \times \dot{\mathbf{R}}^2$ to be

$$\tilde{\alpha}_A = \mathrm{i}(\mathrm{d}x_0 + A). \tag{2.2.3}$$

Owing to the shift property of $A$, $\tilde{\alpha}_A$ is invariant under the $\mathbf{Z} \times {}_\omega \mathbf{Z}^2$ action; $L_{(m_0,\boldsymbol{m})}^* \tilde{\alpha}_A = \tilde{\alpha}_A$. Then $\tilde{\alpha}_A$ projects to a 1-form $\alpha_A$ on $\dot{P}_\omega^3$, so that one has $\Pi_\omega^* \alpha_A = \tilde{\alpha}_A$. Further, by using (2.1.11) and (2.2.3), we can easily show that $\alpha_A$ satisfies

$$\alpha_A\left(\frac{\partial}{\partial x_0}\right) = \mathrm{i} \qquad \text{and} \qquad R_g^* \alpha_A = \alpha_A, \tag{2.2.4}$$

where $x_0$ is regarded as one of the local coordinates of $\dot{P}_\omega^3$. Thus, we see that $\alpha_A$ is a connection form on $\dot{P}_\omega^3$. We denote by $\mathcal{C}(\dot{P}_\omega^3)$ the set of connection forms on $\dot{P}_\omega^3$.

So far we have defined the map

$$\mathcal{A}_\omega(\dot{\mathbf{R}}^2) \longrightarrow \mathcal{C}(\dot{P}_\omega^3); \qquad A \longmapsto \alpha_A. \tag{2.2.5}$$

**Proposition 2.2.** *The map (2.2.5) is a bijection.*

**Proof.** It is easy to show that the map (2.2.5) is injective. We now show that it is surjective as well. For a connection $\alpha \in \mathcal{C}(\dot{P}_\omega^3)$, the pull-back $\Pi_\omega^* \alpha$ takes the form $\mathrm{i}(\mathrm{d}x_0 + A)$, where $A$ is a 1-form expressed as $A = \sum_{k=1}^2 A_k(x_0, \boldsymbol{x})\, \mathrm{d}x_k$ and where we have used the fact that $\alpha(\partial/\partial x_0) = \mathrm{i}$. From (2.1.11) and from $R_g^* \alpha = \alpha$, the form $\Pi_\omega^* \alpha$ turns out to be $L_{(t,\mathbf{0})}$ invariant, so that $A$ proves to be independent of $x_0$, that is, $A$ is viewed as a 1-form on $\dot{\mathbf{R}}^2$; $A = \sum_{k=1}^2 A_k(\boldsymbol{x})\, \mathrm{d}x_k$. Furthermore, from (2.1.10), it follows that $L_{(m_0,\boldsymbol{m})}^* \Pi_\omega^* \alpha = \Pi_\omega^* \alpha$, which results in $\langle \boldsymbol{m}, \omega\, \mathrm{d}\boldsymbol{x}\rangle + T_{\boldsymbol{m}}^* A = A$, so that $A \in \mathcal{A}_\omega(\dot{\mathbf{R}}^2)$. Hence, the map (2.2.5) is surjective. This ends the proof. □

Since $U(1)$ is Abelian, the curvature form of $\alpha \in \mathcal{C}(\dot{P}_\omega^3)$ is defined to be

$$F_\alpha = \mathrm{d}\alpha. \tag{2.2.6}$$

Since $R_g^* F_\alpha = F_\alpha$, $F_\alpha$ defines a 2-form $B$ on $\dot{T}^2$ through

$$F_\alpha = \mathrm{i}\pi_\omega^* B. \tag{2.2.7}$$

$B$ is physically interpreted as a magnetic field on $\dot{T}^2$. As is shown in proposition 2.2, $\alpha$ determines a unique 1-form $A \in \mathcal{A}_\omega(\dot{\mathbf{R}}^2)$ such that $\Pi_\omega^* \alpha = \mathrm{i}(\mathrm{d}x_0 + A)$, so that we have

$$\Pi_\omega^* F_\alpha = \mathrm{i}\, \mathrm{d}A. \tag{2.2.8}$$

We note in addition that since $T_{\boldsymbol{m}}^* \mathrm{d}A = \mathrm{d}A$ from the shift property of $A$, the form $B$ can be viewed as the 2-form $\mathrm{d}A$ through $p^* B = \mathrm{d}A$. Equation (2.2.8) implies that the connection $\alpha$ is flat, i.e., $F_\alpha = 0$, if and only if $\mathrm{d}A = 0$. We denote the set of flat connection forms on $\dot{P}_\omega^3$ by $\mathcal{C}_{\text{flat}}(\dot{P}_\omega^3)$ and the subset of $\mathcal{A}_\omega(\dot{\mathbf{R}}^2)$ consisting of closed 1-forms by

$$\mathcal{Z}_\omega(\dot{\mathbf{R}}^2) = \{A \in \mathcal{A}_\omega(\dot{\mathbf{R}}^2) \mid \mathrm{d}A = 0\}. \tag{2.2.9}$$

**Corollary 2.3.** *The map (2.2.5) induces a bijection of* $\mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$ *to* $\mathcal{C}_{\mathrm{flat}}\big(\dot{P}^3_\omega\big)$.

In the following, we give three examples, which will play a key role in studying the moduli space of flat connections on $\dot{P}^3_\omega$ in section 2.6.

**Example 1** (uniform magnetic fields)**.** We define a 1-form $A$ on $\dot{\mathbf{R}}^2$ to be

$$A = -\langle \boldsymbol{x}, \omega\, \mathrm{d}\boldsymbol{x}\rangle. \tag{2.2.10}$$

It is easy to see that $A \in \mathcal{A}_\omega(\dot{\mathbf{R}}^2)$. Since $\mathrm{d}A = (\omega_{21} - \omega_{12})\, \mathrm{d}x_1 \wedge \mathrm{d}x_2$, $A$ is closed if and only if $\omega$ is symmetric. If we add the term $\langle \varepsilon, \mathrm{d}\boldsymbol{x}\rangle$ with $\varepsilon \in \mathbf{R}^2$, we obtain

$$A = -\langle \boldsymbol{x}, \omega\, \mathrm{d}\boldsymbol{x}\rangle + \langle \varepsilon, \mathrm{d}\boldsymbol{x}\rangle, \tag{2.2.11}$$

which also belongs to $\mathcal{A}_\omega(\dot{\mathbf{R}}^2)$. $\mathrm{d}A$ provides a uniform magnetic field

$$B = (\omega_{21} - \omega_{12})\, \mathrm{d}x_1 \wedge \mathrm{d}x_2. \tag{2.2.12}$$

Tanimura [22] studied connections of this form in the case of $\Lambda = \emptyset$.

**Example 2** (Aharonov–Bohm connections on the 2-torus)**.** The above connection looks of less interest, which can be defined on the whole $\mathbf{R}^2$. In contrast to this, the example given below is of more interest, which is defined not on $\mathbf{R}^2$, but on $\dot{\mathbf{R}}^2$. We make effective use of the Weierstrass zeta function [26], which is given, in a form simplified for our convenience, by

$$\zeta(z) = \frac{1}{z} + \sum_{\boldsymbol{m}\in\mathbf{Z}^2\setminus\{\mathbf{0}\}} \left( \frac{1}{z - m_1 - \mathrm{i}m_2} + \frac{1}{m_1 + \mathrm{i}m_2} + \frac{z}{(m_1 + \mathrm{i}m_2)^2} \right), \tag{2.2.13}$$

where $z \in \mathbf{C}$ and $\boldsymbol{m} = (m_1, m_2)^{\mathrm{T}} \in \mathbf{Z}^2$ with the superscript 'T' indicating the transpose.

**Lemma 2.4.** *$\zeta(z)$ has the following properties.*

*(1)* $\zeta(z)$ *is holomorphic on* $\mathbf{C}\setminus(\mathbf{Z}+\mathrm{i}\mathbf{Z})$ *and has poles of order 1 at* $\mathbf{Z}+\mathrm{i}\mathbf{Z}$. *The residues of* $\zeta$ *at the poles are all equal to 1.* $-\zeta(z)$ *is a primitive function of the Weierstrass* $\wp$*-function;* $\wp(z) = -\zeta'(z)$.
*(2)* $\zeta(-z) = -\zeta(z)$, $\zeta(\mathrm{i}z) = -\mathrm{i}\zeta(z)$.
*(3)* $\zeta(z+1) = \zeta(z) + 2\zeta\left(\frac{1}{2}\right)$, $\zeta(z+\mathrm{i}) = \zeta(z) + 2\zeta\left(\frac{\mathrm{i}}{2}\right)$.
*(4)* $\mathrm{i}\zeta\left(\frac{1}{2}\right) - \zeta\left(\frac{\mathrm{i}}{2}\right) = \pi\mathrm{i}$.
*(5)* $\zeta\left(\frac{1}{2}\right) = \frac{\pi}{2}$, $\zeta\left(\frac{\mathrm{i}}{2}\right) = -\frac{\pi\mathrm{i}}{2}$.

The proof of this lemma is found in a textbook of complex function theory (see [26], for example).

We identify $\mathbf{R}^2$ with $\mathbf{C}$ by $z = x_1 + \mathrm{i}x_2$ and view the set $\Lambda$ as a subset of $\mathbf{C}$, denoting by $c_0^{(j)} \in \mathbf{C}$ the complex numbers corresponding to $\boldsymbol{c}_{\mathbf{0}}^{(j)} \in \mathbf{R}^2$. We define a 1-form $A$ on $\dot{\mathbf{R}}^2$ by

$$A = \frac{1}{2\pi}\, \mathrm{Im} \left( \sum_{j=1}^{N} \nu_j \zeta\big(z - c_0^{(j)}\big)\, \mathrm{d}z \right), \tag{2.2.14}$$

where $\nu_j$ are real parameters. Then, by using lemma 2.4, we can verify that

$$T_{\boldsymbol{m}}^* A = A - \langle \boldsymbol{m}, \tau\, \mathrm{d}\boldsymbol{x}\rangle, \qquad \tau = \begin{pmatrix} 0 & -\frac{1}{2}\sum_{j=1}^{N}\nu_j \\ \frac{1}{2}\sum_{j=1}^{N}\nu_j & 0 \end{pmatrix}, \qquad \boldsymbol{m} \in \mathbf{Z}^2. \tag{2.2.15}$$

On setting $\omega = \tau$ as an anti-symmetric matrix with $\frac{1}{2}\sum \nu_j \in \mathbf{Z}$, namely, on taking the entries of an anti-symmetric matrix $\omega$ to be

$$\omega_{21} = -\omega_{12} = \frac{1}{2}\sum_{j=1}^{N} \nu_j \in \mathbf{Z}, \tag{2.2.16}$$

we find that the 1-form $A$ defined by (2.2.14) belongs to $\mathcal{A}_\omega(\dot{\mathbf{R}}^2)$. Furthermore, the Cauchy–Riemann equations show that $\mathrm{d}A = 0$ on $\dot{\mathbf{R}}^2$, so that $A \in \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$ and hence $\alpha_A \in \mathcal{C}_{\text{flat}}(\dot{P}^3_\omega)$. Though this connection is flat, it may have non-vanishing fluxes at singular points $\boldsymbol{c}_{\mathbf{0}}^{(j)}$, $j = 1, \ldots, N$. In fact, one verifies that

$$\int_{C_\epsilon(\boldsymbol{c}_{\mathbf{0}}^{(j)})} A = \frac{1}{2\pi}\sum_{l=1}^{N} \nu_l \operatorname{Im}\int_{C_\epsilon(c_0^{(j)})} \zeta\left(z - c_0^{(l)}\right)\mathrm{d}z = \nu_j, \tag{2.2.17}$$

where $C_\epsilon\left(\boldsymbol{c}_{\mathbf{0}}^{(j)}\right)$ denotes a circle of radius $\epsilon > 0$ centred at $\boldsymbol{c}_{\mathbf{0}}^{(j)}$ with the assumption that this circle is so small as not to surround any singular points $\boldsymbol{c}_{\mathbf{0}}^{(l)}$, $l \neq j$, other than $\boldsymbol{c}_{\mathbf{0}}^{(j)}$. For this $A$, the connection $\alpha_A$ can be interpreted locally as an Aharonov–Bohm potential. To see this, we take a local section $\sigma : U_j \to \dot{P}^3_\omega$, where $U_j$ is an open subset of $\dot{T}^2$ determined by $p(\boldsymbol{x}) \in U_j \Leftrightarrow 0 < \left|\boldsymbol{x} - \boldsymbol{c}_{\mathbf{0}}^{(j)}\right| < \epsilon'$ and $\epsilon' < \epsilon$. Then, we obtain

$$\sigma^*\alpha_A = \frac{\nu_j}{2\pi}\operatorname{Im}\frac{\mathrm{d}z}{z - c_0^{(j)}} + \frac{1}{2\pi}\operatorname{Im}\left(h_j\left(z - c_0^{(j)}\right)\mathrm{d}z\right), \tag{2.2.18}$$

where $h_j\left(z - c_0^{(j)}\right)$ is a holomorphic function on $U_j \cup \left\{c_0^{(j)}\right\}$. Since $h\left(z - c_0^{(j)}\right)\mathrm{d}z$ is locally an exact form, the first term on the right-hand side of (2.2.18) is the principal part of the gauge potential $\sigma^*\alpha_A$, which is the same as the well-known A-B potential with the singularity at $c_0^{(j)}$. We note in addition that the magnetic field $B$ associated with this connection is expressed as

$$p^*B = \mathrm{d}A = \sum_{j=1}^{N}\sum_{\boldsymbol{m}\in\mathbf{Z}^2} \nu_j\delta\left(\boldsymbol{x} - \boldsymbol{c}_{\boldsymbol{m}}^{(j)}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2, \tag{2.2.19}$$

if we take the exterior differential $\mathrm{d}A$ in the sense of distribution. Arai [5] defined connections with singularity on $\mathbf{R}^2$ and studied them together with the momentum operators in quantum mechanics. He gave an example of singular gauge potential by using the Weierstrass zeta function.

**Example 3** (flat connections looking singular but fluxless). We now take the Weierstrass $\wp$-function [26] in place of the zeta function,

$$\wp(z) = \frac{1}{z^2} + \sum_{\boldsymbol{m}\in\mathbf{Z}^2\setminus\{\mathbf{0}\}}\left(\frac{1}{(z - m_1 - \mathrm{i}m_2)^2} - \frac{1}{(m_1 + \mathrm{i}m_2)^2}\right), \tag{2.2.20}$$

and define the 1-form $A$, like (2.2.14), by

$$A = \frac{1}{2\pi}\operatorname{Im}\left(\sum_{j=1}^{N}\mu_j\wp\left(z - c_0^{(j)}\right)\mathrm{d}z\right). \tag{2.2.21}$$

Since the $\wp$-function is doubly periodic on $\mathbf{R}^2$, we have $T_{\boldsymbol{m}}^*A = A$ for all $\boldsymbol{m} \in \mathbf{Z}^2$. Further, the form $A$ is closed because of the Cauchy–Riemann equations. Thus, we see that $A \in \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$ with $\omega = 0$. However, in contrast with (2.2.17), the fluxes of $A$ at $\boldsymbol{c}_{\mathbf{0}}^{(j)}$, $j = 1, \ldots, N$, vanish,

$$\int_{C_\epsilon\left(\boldsymbol{c}_{\mathbf{0}}^{(j)}\right)} A = \frac{1}{2\pi}\sum_{l=1}^{N}\mu_l\operatorname{Im}\int_{C_\epsilon(c_0^{(j)})}\wp\left(z - c_0^{(l)}\right)\mathrm{d}z = 0. \tag{2.2.22}$$

This is because the points $c_0^{(j)}$ are poles of order 2. We will show in section 2.5 that this connection is gauge equivalent to a vanishing connection $A' = 0$, if the quantization condition, $\frac{1}{2}\sum_{j=1}^{N}\mu_j \in \mathbf{Z}$, like (2.2.16), is satisfied. In fact, under the quantization condition, we will show that there exists a $U(1)$-valued function $f$ on $\dot{T}^2$ such that $A + \frac{1}{2\pi\mathrm{i}}f^{-1}\,\mathrm{d}f = 0$ (see (2.5.21)).

### *2.3. Flux quantization*

As is known from the examples given in the last subsection, the flux at a singularity plays a key role. This subsection shows that the sum of fluxes should be equal to the integer which characterizes the bundle $\dot{P}^3_\omega$.

Since $\Lambda$ has no dense subset, there exists a positive constant $\epsilon$ such that

$$0 < \epsilon < \inf\left\{\left|c^{(j)}_{\boldsymbol{m}} - c^{(l)}_{\boldsymbol{n}}\right| \,\middle|\, c^{(j)}_{\boldsymbol{m}}, c^{(l)}_{\boldsymbol{n}} \in \Lambda, c^{(j)}_{\boldsymbol{m}} \neq c^{(l)}_{\boldsymbol{n}}\right\}. \tag{2.3.1}$$

For $A \in \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$, we define the flux of $A$ at $c^{(j)}_{\boldsymbol{m}}$, $j = 1, \ldots, N$, to be

$$\rho_j(A) = \oint_{C_\epsilon(c^{(j)}_{\boldsymbol{m}})} A. \tag{2.3.2}$$

We have to note here that the right-hand side of (2.3.2) is independent of $\boldsymbol{m} \in \mathbf{Z}^2$ and of $\epsilon$ if it is small enough, as is easily seen from the shift property $T^*_{\boldsymbol{m}} A = A - \langle \boldsymbol{m}, \omega\, \mathrm{d}\boldsymbol{x}\rangle$ and from $\mathrm{d}A = 0$ together with Green's theorem.

**Proposition 2.5.** *For $A \in \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$, the sum of the fluxes is quantized to take an integer value,*

$$\sum_{j=1}^{N} \rho_j(A) = \omega_{21} - \omega_{12}, \tag{2.3.3}$$

*where the number on the right-hand side is characteristic of the bundle $\dot{P}^3_\omega$ (see proposition 2.1).*

**Proof.** Let $D$ be a subset of $\dot{\mathbf{R}}^2$ defined to be

$$D = \{\boldsymbol{x} \in \dot{\mathbf{R}}^2 \mid \boldsymbol{x} + t\boldsymbol{e}_k \in \dot{\mathbf{R}}^2, t \in \mathbf{R}, k = 1, 2\}. \tag{2.3.4}$$

We note here that if $\boldsymbol{x} \in D$ then $\boldsymbol{x} + \boldsymbol{e}_k \in D$, $k = 1, 2$. For a given $\boldsymbol{a} \in D$, we take a closed square path

$$\ell : \boldsymbol{a} \to \boldsymbol{a} + \boldsymbol{e}_1 \to \boldsymbol{a} + \boldsymbol{e}_1 + \boldsymbol{e}_2 \to \boldsymbol{a} + \boldsymbol{e}_2 \to \boldsymbol{a}, \tag{2.3.5}$$

where $\boldsymbol{a} \to \boldsymbol{a}+\boldsymbol{e}_1$, $\boldsymbol{a}+\boldsymbol{e}_1 \to \boldsymbol{a}+\boldsymbol{e}_1+\boldsymbol{e}_2$, etc, denote the line segments from $\boldsymbol{a}$ to $\boldsymbol{a}+\boldsymbol{e}_1$, $\boldsymbol{a}+\boldsymbol{e}_1$ to $\boldsymbol{a} + \boldsymbol{e}_1 + \boldsymbol{e}_2$, etc, respectively. Then, by Green's theorem together with $\mathrm{d}A = 0$ on $\dot{\mathbf{R}}^2$, we obtain

$$\sum_{j=1}^{N} \rho_j(A) = \oint_\ell A. \tag{2.3.6}$$

The right-hand side of the above equation is expressed and calculated as

$$\begin{aligned}
\oint_\ell A &= \int_a^{a+e_1} A + \int_{a+e_1}^{a+e_1+e_2} A - \int_{a+e_2}^{a+e_1+e_2} A - \int_a^{a+e_2} A \\
&= \int_a^{a+e_1} A + \int_a^{a+e_2} T^*_{e_1} A - \int_a^{a+e_1} T^*_{e_2} A - \int_a^{a+e_2} A \\
&= \int_a^{a+e_1} \langle \boldsymbol{e}_2, \omega\, \mathrm{d}\boldsymbol{x}\rangle - \int_a^{a+e_2} \langle \boldsymbol{e}_1, \omega\, \mathrm{d}\boldsymbol{x}\rangle \\
&= \omega_{21} - \omega_{12},
\end{aligned} \tag{2.3.7}$$

where the symbol $\int_a^{a+e_1}$ denotes the integration along the line segment from $\boldsymbol{a}$ to $\boldsymbol{a} + \boldsymbol{e}_1$, and the other integration symbols are interpreted likewise. Thus, equation (2.3.3) proves to hold. □

As the number $\omega_{21} - \omega_{12}$ is characteristic of the bundle $\dot{P}^3_\omega$ and independent of the connections, equation (2.3.3) implies that the sum of fluxes is independent of the choice of connections. On the other hand, this number may be interpreted as the integral of the magnetic field on the whole 2-torus $T^2$, if we take the differentiation in the sense of distribution. In fact, from $p^*B = \mathrm{d}A$, we then have

$$\int_{T^2} B = \int_{I^2} \sum_j \rho_j(A)\delta\big(\boldsymbol{x} - \boldsymbol{c}_{\mathbf{0}}^{(j)}\big)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 = \sum_{j=1}^{N} \rho_j(A). \tag{2.3.8}$$

This means that the sum of fluxes plays the same role as the integral of the (singular) curvature on $T^2$, so that $\omega_{21} - \omega_{12}$ may serve as the first Chern number.

**Proposition 2.6.** *Let $N = 0$. If $\omega \in M(2, \mathbf{Z})$ is not symmetric, then $\mathcal{C}_{\mathrm{flat}}\big(\dot{P}^3_\omega\big) = \emptyset$.*

**Proof.** In the case of $N = 0$, one has $\Lambda = \emptyset$, so that $\dot{\mathbf{R}}^2 = \mathbf{R}^2$. If $\mathcal{C}_{\mathrm{flat}}\big(\dot{P}^3_\omega\big) \neq \emptyset$, and hence $\mathcal{Z}_\omega(\mathbf{R}^2) \neq \emptyset$ because of corollary 2.3, then Green's theorem shows that $\oint_\ell A = 0$ for $A \in \mathcal{Z}_\omega(\mathbf{R}^2)$. On the other hand, equation (2.3.7) still holds true. Hence, it follows that $\omega_{12} = \omega_{21}$, that is, $\omega$ is symmetric. This proves the proposition. □

This proposition implies that proposition 2.5 holds true even if $N = 0$, that is, both sides of (2.3.3) vanish. We now take $A = -\langle \boldsymbol{x}, \omega\,\mathrm{d}\boldsymbol{x}\rangle$, which is not in $\mathcal{Z}_\omega(\mathbf{R}^2)$ if $\omega$ is not symmetric. Then, we obtain

$$\int_{T^2} B = \omega_{21} - \omega_{12}, \tag{2.3.9}$$

which is in comparison with (2.3.3). In the case of $P^3_\omega$ (without singularity), the above quantity is, in fact, the first Chern number.

### *2.4. Gauge group*

A differentiable automorphism $\phi$ of $\dot{P}^3_\omega$ is called a gauge transformation, if it satisfies

$$(\mathrm{GT1})\, R_g \circ \phi = \phi \circ R_g \qquad \text{and} \qquad (\mathrm{GT2})\, \pi_\omega \circ \phi = \pi_\omega. \tag{2.4.1}$$

We denote the set of gauge transformations of $\dot{P}^3_\omega$ by $\mathcal{G}\big(\dot{P}^3_\omega\big)$, which becomes a group with the composition of maps as the multiplication operation, and is called a gauge group.

Now we denote the set of $U(1)$-valued $C^\infty$-functions on $\dot{T}^2$ by

$$C^\infty(\dot{T}^2; U(1)) = \{h : \dot{\mathbf{R}}^2 \to U(1) \mid T^*_{\boldsymbol{m}} h = h, \boldsymbol{m} \in \mathbf{Z}^2\}. \tag{2.4.2}$$

$C^\infty(\dot{T}^2; U(1))$ is made into an Abelian group with respect to the multiplication operation defined for $h_1, h_2 \in C^\infty(\dot{T}^2; U(1))$ through

$$(h_1 h_2)(\boldsymbol{x}) = h_1(\boldsymbol{x}) h_2(\boldsymbol{x}), \qquad \boldsymbol{x} \in \dot{\mathbf{R}}^2. \tag{2.4.3}$$

For $h \in C^\infty(\dot{T}^2; U(1))$, we define a map $\phi_h : \dot{P}^3_\omega \to \dot{P}^3_\omega$ through

$$\phi_h([(x_0, \boldsymbol{x})]) = [(x_0, \boldsymbol{x})] \cdot h(\boldsymbol{x}), \tag{2.4.4}$$

which is well defined because of the periodicity of $h$, $T^*_{\boldsymbol{m}} h = h$. Since $U(1)$ is Abelian and since $\pi_\omega \circ R_{h(\boldsymbol{x})} = \pi_\omega$, one has $\phi_h \circ R_g = R_g \circ \phi_h$ and $\pi_\omega \circ \phi_h = \pi_\omega$, respectively. Thus, we have shown that $\phi_h \in \mathcal{G}\big(\dot{P}^3_\omega\big)$.

**Proposition 2.7.** *The map*

$$C^\infty(\dot{T}^2; U(1)) \longrightarrow \mathcal{G}\big(\dot{P}^3_\omega\big) : h \longmapsto \phi_h \tag{2.4.5}$$

*is a group isomorphism.*

**Proof.** Clearly, the map (2.4.5) is a homomorphism. We first prove that the map (2.4.5) is injective. Assume that $\phi_h = \mathrm{id}_{\mathcal{G}(\dot{P}^3_\omega)}$ for $h \in C^\infty(\dot{T}^2; U(1))$, where $\mathrm{id}_{\mathcal{G}(\dot{P}^3_\omega)}$ denotes the identity in $\mathcal{G}(\dot{P}^3_\omega)$. We may express $h(\boldsymbol{x})$ as $h(\boldsymbol{x}) = \mathrm{e}^{2\pi \mathrm{i}\theta(\boldsymbol{x})}$, where $\theta(\boldsymbol{x})$ is a real-valued function determined up to additional integers. Then $\phi_h = \mathrm{id}_{\mathcal{G}(\dot{P}^3_\omega)}$ implies that $[(x_0 + \theta(\boldsymbol{x}), \boldsymbol{x})] = [(x_0, \boldsymbol{x})]$ for any $(x_0, \boldsymbol{x}) \in \mathbf{R} \times \dot{\mathbf{R}}^2$. It then follows that $\theta(\boldsymbol{x}) \in \mathbf{Z}$, hence $h(\boldsymbol{x}) = 1$ for any $\boldsymbol{x} \in \dot{\mathbf{R}}^2$. This proves the injectivity of (2.4.5). We now prove that the map (2.4.5) is also surjective. For $\phi \in \mathcal{G}(\dot{P}^3_\omega)$, the property (GT2) implies that there exists a $U(1)$-valued function $\widetilde{h}$ on $\mathbf{R} \times \dot{\mathbf{R}}^2$ such that

$$(\phi \circ \Pi_\omega)(x_0, \boldsymbol{x}) = \Pi_\omega(x_0, \boldsymbol{x}) \cdot \widetilde{h}(x_0, \boldsymbol{x}). \tag{2.4.6}$$

Operating the above equation with $R_g$, $g = \mathrm{e}^{2\pi \mathrm{i}t}$, one has

$$(R_g \circ \phi \circ \Pi_\omega)(x_0, \boldsymbol{x}) = \Pi_\omega(x_0, \boldsymbol{x}) \cdot \widetilde{h}(x_0, \boldsymbol{x})\, \mathrm{e}^{2\pi \mathrm{i}t}. \tag{2.4.7}$$

Owing to (GT1) and (2.1.11), the left-hand side of (2.4.7) is brought into the form

$$\begin{aligned}(\phi \circ R_g \circ \Pi_\omega)(x_0, \boldsymbol{x}) &= (\phi \circ \Pi_\omega \circ L_{(t,\mathbf{0})})(x_0, \boldsymbol{x}) \\ &= \Pi_\omega(L_{(t,\mathbf{0})}(x_0, \boldsymbol{x})) \cdot \widetilde{h}(L_{(t,\mathbf{0})}(x_0, \boldsymbol{x})) \\ &= \Pi_\omega(x_0 + t, \boldsymbol{x}) \cdot \widetilde{h}(x_0 + t, \boldsymbol{x}) \\ &= \Pi_\omega(x_0, \boldsymbol{x}) \cdot \mathrm{e}^{2\pi \mathrm{i}t} \widetilde{h}(x_0 + t, \boldsymbol{x}).\end{aligned} \tag{2.4.8}$$

Equations (2.4.7) and (2.4.8) are put together to imply that $\widetilde{h}(x_0 + t, \boldsymbol{x}) = \widetilde{h}(x_0, \boldsymbol{x})$, so that $\widetilde{h}$ is independent of $x_0$ and becomes a function $h$ on $\dot{\mathbf{R}}^2$; $\widetilde{h}(x_0, \boldsymbol{x}) = h(\boldsymbol{x})$. We now show that $h$ is $T_{\boldsymbol{m}}$ invariant. Composed with $\phi$ to the left, equation (2.1.10) provides $\phi \circ \Pi_\omega \circ L_{(m_0,\boldsymbol{m})} = \phi \circ \Pi_\omega$. The left- and right-hand sides are put, respectively, in the form

$$(\phi \circ \Pi_\omega \circ L_{(m_0,\boldsymbol{m})})(x_0, \boldsymbol{x}) = \Pi_\omega(x_0, \boldsymbol{x}) \cdot h(\boldsymbol{x} + \boldsymbol{m}), \tag{2.4.9}$$

$$\phi \circ \Pi_\omega(x_0, \boldsymbol{x}) = \Pi_\omega(x_0, \boldsymbol{x}) \cdot h(\boldsymbol{x}). \tag{2.4.10}$$

xIt then follows that $h(\boldsymbol{x} + \boldsymbol{m}) = h(\boldsymbol{x})$ for $\boldsymbol{x} \in \dot{\mathbf{R}}^2$. Hence, $h \in C^\infty(\dot{T}^2; U(1))$. Thus, the map (2.4.5) is also surjective. This ends the proof. □

### 2.5. *Gauge equivalence*

The gauge group $\mathcal{G}(\dot{P}^3_\omega)$ acts on the set $\mathcal{C}(\dot{P}^3_\omega)$ of connections in the natural manner,

$$\mathcal{G}(\dot{P}^3_\omega) \times \mathcal{C}(\dot{P}^3_\omega) \longrightarrow \mathcal{C}(\dot{P}^3_\omega) : (\phi, \alpha) \longmapsto \phi^* \alpha. \tag{2.5.1}$$

It then defines an equivalence relation on $\mathcal{C}(\dot{P}^3_\omega)$ through

$$\alpha \sim \alpha' \quad \Longleftrightarrow \quad \exists \phi \in \mathcal{G}(\dot{P}^3_\omega) \qquad \text{s.t. } \alpha' = \phi^* \alpha. \tag{2.5.2}$$

The connections $\alpha$ and $\alpha'$ subject to $\alpha \sim \alpha'$ are called gauge equivalent. Our interest, however, will centre on the gauge equivalence restricted on $\mathcal{C}_{\mathrm{flat}}(\dot{P}^3_\omega)$. Owing to corollary 2.3 and proposition 5, we may discuss the gauge equivalence in $\mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$ with respect to the action of $C^\infty(\dot{T}^2; U(1))$.

**Lemma 2.8.** *For $A, A' \in \mathcal{A}_\omega(\dot{\mathbf{R}}^2)$ and for $\alpha_A, \alpha_{A'} \in \mathcal{C}(\dot{P}^3_\omega)$, one has*

$$\alpha_{A'} = \phi_h^* \alpha_A \quad \Longleftrightarrow \quad A' = A + \frac{1}{2\pi \mathrm{i}} h^{-1}\, \mathrm{d}h, \qquad h \in C^\infty(\dot{T}^2; U(1)), \tag{2.5.3}$$

*where $h^{-1}$ denotes the inverse of $h$ as an element of the group $C^\infty(\dot{T}^2; U(1))$.*

**Proof.** We express $h \in C^\infty(\dot{T}^2; U(1))$ as $h(\boldsymbol{x}) = \mathrm{e}^{2\pi \mathrm{i}\theta(\boldsymbol{x})}$, where $\theta(\boldsymbol{x})$ is a real-valued function determined modulo $\mathbf{Z}$. Then we have

$$\Pi_\omega^* \alpha_{A'} = \mathrm{i}(\mathrm{d}x_0 + A'), \tag{2.5.4}$$

$$\Pi_\omega^* \phi_h^* \alpha_A = \mathrm{i}(\mathrm{d}x_0 + \mathrm{d}\theta + A) = \mathrm{i}\left(\mathrm{d}x_0 + A + \frac{1}{2\pi\mathrm{i}} h^{-1}\,\mathrm{d}h\right). \tag{2.5.5}$$

These equations are put together to result in (2.5.3). This ends the proof. □

We are now in a position to prove our main theorem which gives necessary and sufficient conditions for two flat connections to be gauge equivalent. To state the theorem, it is convenient to introduce the following quantities in addition to the fluxes given in (2.3.2); for $\boldsymbol{a} \in D$, we define the quantity $p_k(\boldsymbol{a}, A)$ to be

$$p_k(\boldsymbol{a}, A) = \int_{I_k(\boldsymbol{a})} A, \qquad k = 1, 2, \tag{2.5.6}$$

where $I_k(\boldsymbol{a})$ denotes the line segment from $\boldsymbol{a}$ to $\boldsymbol{a} + \boldsymbol{e}_k$.

**Theorem 2.9.** *Let $A, A' \in \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$ and $\alpha_A, \alpha_{A'} \in \mathcal{C}_{\mathrm{flat}}(\dot{P}^3_\omega)$ (see corollary 2.3). The flat connections $\alpha_A$ and $\alpha_{A'}$ are gauge equivalent, if and only if they satisfy*

$$\mathrm{e}^{2\pi\mathrm{i}\rho_j(A)} = \mathrm{e}^{2\pi\mathrm{i}\rho_j(A')}, \qquad j = 1, \ldots, N, \tag{2.5.7}$$

*and*

$$\mathrm{e}^{2\pi\mathrm{i}p_k(\boldsymbol{a},A)} = \mathrm{e}^{2\pi\mathrm{i}p_k(\boldsymbol{a},A')}, \qquad k = 1, 2, \tag{2.5.8}$$

*where $\boldsymbol{a} \in D$.*

**Proof.** Let $\alpha_A$ and $\alpha_{A'}$ be gauge equivalent. Then, according to lemma 2.8, there exists $h \in C^\infty(\dot{T}^2; U(1))$ such that $A$ and $A'$ are related as on the right-hand side of (2.5.3). We now denote by $h_\epsilon$ the restriction of $h$ to the circle $C_\epsilon\big(\boldsymbol{c}_0^{(j)}\big) = \{\boldsymbol{x} \in \dot{\mathbf{R}}^2 \,\big|\, \big|\boldsymbol{x} - \boldsymbol{c}_0^{(j)}\big| = \epsilon\}$. Then, $h_\epsilon$ is viewed as a map from this circle to $U(1)$. We note here that the normalized volume element of $U(1)$ is given by $\frac{1}{2\pi\mathrm{i}} g^{-1}\,\mathrm{d}g$, $g \in U(1)$. Then, we verify that

$$\oint_{C_\epsilon\left(\boldsymbol{c}_0^{(j)}\right)} \frac{1}{2\pi\mathrm{i}} h^{-1}\,\mathrm{d}h = \oint_{C_\epsilon\left(\boldsymbol{c}_0^{(j)}\right)} h_\epsilon^*\left(\frac{1}{2\pi\mathrm{i}} g^{-1}\,\mathrm{d}g\right) = \deg h_\epsilon, \tag{2.5.9}$$

where $\deg h_\epsilon$ denotes the degree of the map $h_\epsilon : C_\epsilon\big(\boldsymbol{c}_0^{(j)}\big) \cong S^1 \to U(1)$, which takes an integer value. From (2.5.3) and (2.5.9), it follows that

$$\begin{aligned}\mathrm{e}^{2\pi\mathrm{i}\rho_j(A')} &= \exp\left(2\pi\mathrm{i}\oint_{C_\epsilon(\boldsymbol{c}_0^{(j)})}\left(A + \frac{1}{2\pi\mathrm{i}} h^{-1}\,\mathrm{d}h\right)\right)\\ &= \mathrm{e}^{2\pi\mathrm{i}(\rho_j(A) + \deg h_\epsilon)}\\ &= \mathrm{e}^{2\pi\mathrm{i}\rho_j(A)}.\end{aligned} \tag{2.5.10}$$

We now take an arbitrary point $\boldsymbol{a} \in D$. Since $T_{\boldsymbol{e}_k}^* h = h$, the restriction of $h$ on the line segment $I_k(\boldsymbol{a})$ may be viewed as a map $S^1$ to $U(1)$. Then, like (2.5.10), we obtain

$$\mathrm{e}^{2\pi\mathrm{i}p_k(\boldsymbol{a},A')} = \exp\left(2\pi\mathrm{i}\int_{I_k(\boldsymbol{a})}\left(A + \frac{1}{2\pi\mathrm{i}} h^{-1}\,\mathrm{d}h\right)\right) = \mathrm{e}^{2\pi\mathrm{i}p_k(\boldsymbol{a},A)}. \tag{2.5.11}$$

Thus, equations (2.5.7) and (2.5.8) prove to be necessary conditions.

Conversely, we assume that equations (2.5.7) and (2.5.8) are satisfied. From (2.5.7) together with $\mathrm{d}A = \mathrm{d}A' = 0$ on $\dot{\mathbf{R}}^2$, it follows that for any $\boldsymbol{x} \in \dot{\mathbf{R}}^2$, the quantity $\exp\big(2\pi\mathrm{i}\int_{\boldsymbol{x}}^{\boldsymbol{x}+\boldsymbol{e}_k}(A - A')\big)$ is defined independently of the choice of paths joining $\boldsymbol{x}$ to $\boldsymbol{x} + \boldsymbol{e}_k$, so that this quantity determines a function on $\dot{\mathbf{R}}^2$. Furthermore, since one has $T_{\boldsymbol{e}_k}^*(A - A') = A - A'$

on account of the shift property of $A$ and $A'$, the differential of $\exp\left(2\pi\mathrm{i}\int_{x}^{x+e_k}(A-A')\right)$ turns out to be evaluated as

$$\begin{aligned}\mathrm{d}\exp&\left(2\pi\mathrm{i}\int_{x}^{x+e_k}(A-A')\right)\\&=2\pi\mathrm{i}\left(T_{e_k}^{*}(A-A')-(A-A')\right)\exp\left(2\pi\mathrm{i}\int_{x}^{x+e_k}(A-A')\right)=0.\end{aligned}\tag{2.5.12}$$

This and equation (2.5.8) are put together to imply that

$$\exp\left(2\pi\mathrm{i}\int_{x}^{x+e_k}(A-A')\right)=1\tag{2.5.13}$$

for any $x\in\dot{\mathbf{R}}^2$.

We define a function $h:\dot{\mathbf{R}}^2\to U(1)$ to be

$$h(x)=\exp\left(2\pi\mathrm{i}\int_{a}^{x}(A-A')\right),\tag{2.5.14}$$

where $a\in\dot{\mathbf{R}}^2$ is the point assigned in equation (2.5.8). Note that the right-hand side of the above equation is independent of the choice of paths on account of (2.5.7) and of $\mathrm{d}(A'-A)=0$ on $\dot{\mathbf{R}}^2$, so that it defines a function on $\dot{\mathbf{R}}^2$. From (2.5.13), we verify that

$$\begin{aligned}h(x+e_k)&=\exp\left(2\pi\mathrm{i}\int_{a}^{x+e_k}(A-A')\right)\\&=\exp\left(2\pi\mathrm{i}\int_{a}^{x}(A-A')\right)\exp\left(2\pi\mathrm{i}\int_{x}^{x+e_k}(A-A')\right)\\&=h(x).\end{aligned}\tag{2.5.15}$$

This implies that $h\in C^\infty(\dot{T}^2;U(1))$. The function $h$ is differentiated to give

$$\frac{1}{2\pi\mathrm{i}}h^{-1}\,\mathrm{d}h=A'-A,\tag{2.5.16}$$

which shows that $\alpha_A$ and $\alpha_{A'}$ are gauge equivalent on account of (2.5.3). This completes the proof. □

**Remark.** In the course of the proof, we have shown that the property (2.5.8) holds for any $a\in\dot{\mathbf{R}}^2$ if $\alpha_A$ and $\alpha_{A'}$ are gauge equivalent.

In conclusion, we apply lemma 2.8 to show that the connection $A$ given in example 3 is gauge equivalent to $A'=0$, if the quantization condition to be stated below is satisfied. According to (2.5.14) with $A'=0$, we define a function $f:\dot{\mathbf{R}}^2\to U(1)$ to be

$$f(x)=\exp\left(-2\pi\mathrm{i}\int_{a}^{x}A\right).\tag{2.5.17}$$

Note here that the integral on the right-hand side of the above equation is defined independently of the choice of paths joining $a$ and $x\in\dot{\mathbf{R}}^2$ on account of $\mathrm{d}A=0$ on $\dot{\mathbf{R}}^2$ and of the fact that the flux at every point of $\Lambda$ vanishes (see (2.2.22)). The function $f$ is actually evaluated as

$$f(x)=\exp\left(\mathrm{i}\sum_{j=1}^{N}\mu_j\,\mathrm{Im}\big(\zeta\big(z-c_0^{(j)}\big)-\zeta\big(a-c_0^{(j)}\big)\big)\right),\tag{2.5.18}$$

where $a$ is the complex number corresponding to $a\in\mathbf{R}^2$. Further, by using lemma 2.4, we can verify that

$$f(x+e_1)=f(x),\qquad f(x+e_2)=f(x)\exp\left(-\pi\mathrm{i}\sum_{j=1}^{N}\mu_j\right).\tag{2.5.19}$$

Hence, if the quantization condition

$$\frac{1}{2}\sum_{j=1}^{N}\mu_j \in \mathbf{Z} \tag{2.5.20}$$

is satisfied, $f$ becomes doubly periodic, so that $f \in C^\infty(\dot{T}^2; U(1))$. For this function, we obtain

$$\frac{1}{2\pi \mathrm{i}} f^{-1}\,\mathrm{d}f = \frac{-1}{2\pi}\sum_{j=1}^{N}\mu_j \,\mathrm{Im}\big(\wp\big(z - c_0^{(j)}\big)\,\mathrm{d}z\big) = -A. \tag{2.5.21}$$

Hence, lemma 2.8 together with the above equation implies that $A$ is gauge equivalent to $A' = 0$.

### *2.6. Moduli space of flat connections*

So far we have studied the gauge equivalence of flat connections. We now discuss $\mathcal{C}_{\text{flat}}\big(\dot{P}^3_\omega\big)/\mathcal{G}\big(\dot{P}^3_\omega\big)$, the set of gauge-inequivalent flat connections. Proposition 2.5 and theorem 2.9 are put together to show that the $N+1$ quantities $\mathrm{e}^{2\pi \mathrm{i}\rho_j(A)}$, $j = 1, \ldots, N-1$, and $\mathrm{e}^{2\pi\mathrm{i} p_k(\boldsymbol{a},A)}$, $k = 1, 2$, determine inequivalent flat connections, where $\boldsymbol{a}$ is an arbitrarily chosen point of $D$.

**Theorem 2.10.** *For $N > 0$, the map $\mathcal{Z}_\omega(\dot{\mathbf{R}}^2) \to T^{N+1}$ defined by*

$$A \mapsto \big(\mathrm{e}^{2\pi\mathrm{i}\rho_1(A)}, \ldots, \mathrm{e}^{2\pi\mathrm{i}\rho_{N-1}(A)}, \mathrm{e}^{2\pi\mathrm{i}p_1(\boldsymbol{a},A)}, \mathrm{e}^{2\pi\mathrm{i}p_2(\boldsymbol{a},A)}\big) \tag{2.6.1}$$

*with $\boldsymbol{a}$ an arbitrarily chosen point of $D$ gives rise to a bijection from the moduli space $\mathcal{C}_{\text{flat}}\big(\dot{P}^3_\omega\big)/\mathcal{G}\big(\dot{P}^3_\omega\big)$ to the $(N+1)$-torus $T^{N+1}$,*

$$\Psi : \mathcal{C}_{\text{flat}}\big(\dot{P}^3_\omega\big)/\mathcal{G}\big(\dot{P}^3_\omega\big) \cong \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)/C^\infty(\dot{T}^2; U(1)) \longrightarrow T^{N+1}. \tag{2.6.2}$$

**Proof.** From theorem 2.9, it follows that the map $\Psi$ is injective. We now show that $\Psi$ is surjective. For an arbitrarily chosen point, $(\mathrm{e}^{2\pi\mathrm{i}t_1}, \ldots, \mathrm{e}^{2\pi\mathrm{i}t_{N-1}}, \mathrm{e}^{2\pi\mathrm{i}\tau_1}, \mathrm{e}^{2\pi\mathrm{i}\tau_2})$, of $T^{N+1}$, we wish to construct a 1-form $A \in \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$ satisfying

$$\mathrm{e}^{2\pi\mathrm{i}\rho_j(A)} = \mathrm{e}^{2\pi\mathrm{i}t_j}, \quad j = 1, \ldots, N-1, \qquad \mathrm{e}^{2\pi\mathrm{i}p_k(\boldsymbol{a},A)} = \mathrm{e}^{2\pi\mathrm{i}\tau_k}, \quad k = 1, 2. \tag{2.6.3}$$

If such an $A$ is found, then $\Psi$ becomes surjective.

First, we break up the matrix $\omega \in M(2, \mathbf{Z})$ into the sum of the symmetric and the anti-symmetric parts,

$$\omega = \omega^s + \omega^a, \qquad \omega^s = \tfrac{1}{2}(\omega + \omega^{\mathrm{T}}), \qquad \omega^a = \tfrac{1}{2}(\omega - \omega^{\mathrm{T}}). \tag{2.6.4}$$

Though the entries of $\omega$ are integers, the entries of $\omega^s$ and $\omega^a$ do not have to be integers. We define a 1-form $A^s$ to be

$$A^s = -\langle \boldsymbol{x}, \omega^s\,\mathrm{d}\boldsymbol{x}\rangle. \tag{2.6.5}$$

Further, by using the Weierstrass zeta function, we define a 1-form $A^a$ to be

$$A^a = \frac{1}{2\pi}\,\mathrm{Im}\left(\sum_{j=1}^{N}\nu_j \zeta\big(z - c_0^{(j)}\big)\,\mathrm{d}z\right), \tag{2.6.6}$$

where the real parameters $\nu_j$ are required to satisfy

$$\frac{1}{2}\sum_{j=1}^{N}\nu_j = \frac{1}{2}(\omega_{21} - \omega_{12}). \tag{2.6.7}$$

We further define a 1-form $A$ by

$$A = A^s + A^a + \langle \varepsilon, \mathrm{d}\boldsymbol{x}\rangle, \tag{2.6.8}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2)^{\mathrm{T}} \in \mathbf{R}^2$ is an undetermined vector. By using (2.2.15), we verify that

$$\begin{aligned} T^*_{\boldsymbol{m}} A &= (A^s - \langle \boldsymbol{m}, \omega^s\, \mathrm{d}\boldsymbol{x}\rangle) + (A^a - \langle \boldsymbol{m}, \omega^a\, \mathrm{d}\boldsymbol{x}\rangle) + \langle \varepsilon, \mathrm{d}\boldsymbol{x}\rangle \\ &= A - \langle \boldsymbol{m}, \omega\, \mathrm{d}\boldsymbol{x}\rangle. \end{aligned} \tag{2.6.9}$$

Moreover, it is easy to see that $\mathrm{d}A = 0$. Thus, we see that $A \in \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$.

We proceed to calculate the fluxes of $A$ at singular points $\boldsymbol{c}_{\mathbf{0}}^{(j)}$. Using Green's theorem and the residue theorem, we obtain

$$\begin{aligned} \rho_j(A) &= \oint_{C_\epsilon(\boldsymbol{c}_{\mathbf{0}}^{(j)})} (-\langle \boldsymbol{x}, \omega^s\, \mathrm{d}\boldsymbol{x}\rangle + A^a + \langle \varepsilon, \mathrm{d}\boldsymbol{x}\rangle) \\ &= \oint_{C_\epsilon(\boldsymbol{c}_{\mathbf{0}}^{(j)})} \mathrm{d}\left(\langle \varepsilon, \boldsymbol{x}\rangle - \frac{1}{2}\langle \boldsymbol{x}, \omega^s \boldsymbol{x}\rangle\right) + \frac{1}{2}\sum_{l=1}^{N} \nu_l \,\mathrm{Im} \oint_{C_\epsilon(c_0^{(j)})} \zeta\left(z - c_0^{(l)}\right) \mathrm{d}z \\ &= \nu_j. \end{aligned} \tag{2.6.10}$$

Further, the quantities $p_k(\boldsymbol{a}, A)$ are put in the form

$$p_k(\boldsymbol{a}, A) = \int_{I_k(\boldsymbol{a})} A = \varepsilon_k + p_k(\boldsymbol{a}, A^s + A^a). \tag{2.6.11}$$

We then choose $\nu_j$, $j = 1, \ldots, N$, and $\varepsilon_k$, $k = 1, 2$, so as to satisfy

$$\mathrm{e}^{2\pi \mathrm{i} \nu_j} = \mathrm{e}^{2\pi \mathrm{i} t_j}, \qquad j = 1, \ldots, N-1, \tag{2.6.12}$$

$$\begin{aligned} \nu_N &= \omega_{21} - \omega_{12} - \sum_{j=1}^{N-1} \nu_j, \\ \mathrm{e}^{2\pi \mathrm{i} \varepsilon_k} &= \mathrm{e}^{2\pi \mathrm{i}(\tau_k - p_k(\boldsymbol{a}, A^s + A^a))}, \qquad k = 1, 2. \end{aligned} \tag{2.6.13}$$

Then we find that the 1-form (2.6.8) with these parameters, $\nu_j$ and $\varepsilon_k$, satisfies equations (2.6.3). This completes the proof. □

The connection $A$ we have constructed above is not the only one that satisfies equation (2.6.3). We here denote by $A^{(\wp)}$ the connection given in example 3 with the quantization condition (2.5.20). Then we can verify that the connection $A + A^{(\wp)}$ also satisfies the same equation. In fact, we have

$$\rho_j(A^{(\wp)}) = 0, \qquad p_1(\boldsymbol{a}, A^{(\wp)}) = 0, \qquad p_2(\boldsymbol{a}, A^{(\wp)}) = \frac{1}{2}\sum_{j=1}^{N} \mu_j \in \mathbf{Z}, \tag{2.6.14}$$

so that $\mathrm{e}^{2\pi \mathrm{i} \rho_j(A + A^{(\wp)})} = \mathrm{e}^{2\pi \mathrm{i} \rho_j(A)}$, $\mathrm{e}^{2\pi \mathrm{i} p_k(\boldsymbol{a}, A + A^{(\wp)})} = \mathrm{e}^{2\pi \mathrm{i} p_k(\boldsymbol{a}, A)}$. In the course of the proof of the above theorem, we have shown the following.

**Theorem 2.11.** *Let $N > 0$ and $A \in \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$. Then there exist $\nu_j \in \mathbf{R}$, $j = 1, \ldots, N$, $\varepsilon \in \mathbf{R}^2$ and $h \in C^\infty(\dot{T}^2; U(1))$ such that*

$$A = A^a - \left\langle \boldsymbol{x}, \frac{\omega + \omega^{\mathrm{T}}}{2}\, \mathrm{d}\boldsymbol{x}\right\rangle + \langle \varepsilon, \mathrm{d}\boldsymbol{x}\rangle + \frac{1}{2\pi \mathrm{i}} h^{-1}\, \mathrm{d}h, \tag{2.6.15}$$

$$A^a = \frac{1}{2\pi}\sum_{j=1}^{N} \nu_j \,\mathrm{Im}\left(\zeta\left(z - c_0^{(j)}\right) \mathrm{d}z\right), \tag{2.6.16}$$

*with*

$$\rho_j(A) = \nu_j, \qquad j = 1, \ldots, N. \tag{2.6.17}$$

So far we have realized the inequivalent flat connections on $\dot{P}^3_\omega$ with non-empty set of singularity. If there is no singularity, non-trivial flat connections exist only when $\omega$ is symmetric. In this case, the map (2.6.1) reduces to $A \mapsto (\mathrm{e}^{2\pi \mathrm{i} p_1(\boldsymbol{a},A)}, \mathrm{e}^{2\pi \mathrm{i} p_2(\boldsymbol{a},A)})$. The following corollary results from this map.

**Corollary 2.12.** *Let $N = 0$, and let $\omega \in M(2, \mathbf{Z})$ be a symmetric matrix. Then, the moduli space $\mathcal{C}_{\mathrm{flat}}(P^3_\omega)/\mathcal{G}(P^3_\omega)$ is identified with $T^2$.*

**Proof.** It suffices for us to prove that the map $\mathcal{Z}_\omega(P^3_\omega) \to T^2$ is surjective. We take $A = -\langle \boldsymbol{x}, \omega\, \mathrm{d}\boldsymbol{x}\rangle + \langle \varepsilon, \mathrm{d}\boldsymbol{x}\rangle$ with $\varepsilon = (\varepsilon_1, \varepsilon_2)^{\mathrm{T}} \in \mathbf{R}^2$ undetermined. For a given $(\mathrm{e}^{2\pi \mathrm{i}\tau_1}, \mathrm{e}^{2\pi \mathrm{i}\tau_2}) \in T^2$, we choose $\varepsilon_k$ so as to satisfy

$$\mathrm{e}^{2\pi \mathrm{i}\varepsilon_k} = \mathrm{e}^{2\pi \mathrm{i}(\tau_k - p_k(\boldsymbol{a}, -\langle \boldsymbol{x}, \omega\, \mathrm{d}\boldsymbol{x}\rangle))}, \qquad k = 1, 2. \tag{2.6.18}$$

Then it turns out that $\mathrm{e}^{2\pi \mathrm{i} p_k(\boldsymbol{a},A)} = \mathrm{e}^{2\pi \mathrm{i}\tau_k}$, $k = 1, 2$. This ends the proof. □

### *2.7. Holonomy*

In this section, we show that conditions (2.5.7) and (2.5.8) in theorem 2.9 mean that the connections $\alpha_A$ and $\alpha_{A'}$ have the same holonomies associated with the closed curves $C_\epsilon(\boldsymbol{c}^{(j)}_{\mathbf{0}})$ and with the cycles $p(I_k(\boldsymbol{x}))$, respectively, on the punctured 2-torus $\dot{T}^2$.

For a curve $(x_0(t), \boldsymbol{x}(t))$ in $\mathbf{R} \times \dot{\mathbf{R}}^2$, we set $c(t) = \Pi_\omega(x_0(t), \boldsymbol{x}(t))$. Then, one has $(\alpha_A)_{c(t)}(\dot{c}(t)) = \Pi^*_\omega \alpha_A(\dot{x}_0(t), \dot{\boldsymbol{x}}(t))$, so that the curve $c(t)$ is horizontal, if and only if $\Pi^*_\omega \alpha_A(\dot{x}_0(t), \dot{\boldsymbol{x}}(t)) = 0$. Hence, we obtain the condition for the curve $c(t)$ to be horizontal, in the form

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} + A(\dot{\boldsymbol{x}}(t)) = 0. \tag{2.7.1}$$

If we are given a point $(a_0, \boldsymbol{a}) \in \mathbf{R} \times \dot{\mathbf{R}}^2$ and a cycle $p(I_k(\boldsymbol{a}))$ in $\dot{T}^2$ with $\boldsymbol{a} \in D$, the above equation is easily integrated to give a curve

$$(x_0(t), \boldsymbol{x}(t)) = \left(a_0 - \int_{\boldsymbol{a}}^{\boldsymbol{a}+t\boldsymbol{e}_k} A, \boldsymbol{a} + t\boldsymbol{e}_k\right), \tag{2.7.2}$$

which projects to a horizontal curve $[(x_0(t), \boldsymbol{x}(t))]$ in $\dot{P}^3_\omega$ with the initial point $[(a_0, \boldsymbol{a})]$. The final point of this curve is related to the initial point as follows:

$$\begin{aligned}\left[\left(a_0 - \int_{I_k(\boldsymbol{a})} A, \boldsymbol{a} + \boldsymbol{e}_k\right)\right] &= [(a_0, \boldsymbol{a} + \boldsymbol{e}_k)] \cdot \exp\left(-2\pi \mathrm{i} \int_{I_k(\boldsymbol{a})} A\right) \\ &= [(-\langle \boldsymbol{e}_k, \omega \boldsymbol{a}\rangle, \mathbf{0}) \cdot (0, \boldsymbol{e}_k) \cdot (a_0, \boldsymbol{a})] \cdot \exp\left(-2\pi \mathrm{i} \int_{I_k(\boldsymbol{a})} A\right) \\ &= [(a_0, \boldsymbol{a})] \cdot \exp\left(-2\pi \mathrm{i}\langle \boldsymbol{e}_k, \omega \boldsymbol{a}\rangle - 2\pi \mathrm{i} \int_{I_k(\boldsymbol{a})} A\right).\end{aligned} \tag{2.7.3}$$

This implies that the holonomy of the cycle $p(I_k(\boldsymbol{a}))$ with respect to $[(a_0, \boldsymbol{a})]$ is given by $\exp(-2\pi \mathrm{i}\langle \boldsymbol{e}_k, \omega \boldsymbol{a}\rangle - 2\pi \mathrm{i} p_k(\boldsymbol{a}, A))$. To be precise, we have to verify that this quantity is independent of the choice of representatives of $[(a_0, \boldsymbol{a})]$. To this end, we assume that $[(a'_0, \boldsymbol{a}')] = [(a_0, \boldsymbol{a})]$. Then, there exists $(m_0, \boldsymbol{m}) \in \mathbf{Z} \times {}_\omega\mathbf{Z}^2$ such that $L_{(m_0, \boldsymbol{m})}(a_0, \boldsymbol{a}) = (a'_0, \boldsymbol{a}')$. By using this, the holonomy with the reference point $[(a'_0, \boldsymbol{a}')]$ is expressed and calculated as

$$\exp\left(-2\pi \mathrm{i}\langle \boldsymbol{e}_k, \omega \boldsymbol{a}'\rangle - 2\pi \mathrm{i}\int_{I_k(\boldsymbol{a}')} A\right) = \exp\left(-2\pi \mathrm{i}(\langle \boldsymbol{e}_k, \omega \boldsymbol{a}\rangle + \langle \boldsymbol{e}_k, \omega \boldsymbol{m}\rangle) - 2\pi \mathrm{i}\int_{I_k(\boldsymbol{a})} T_{\boldsymbol{m}}^* A\right)$$
$$= \exp\left(-2\pi \mathrm{i}\langle \boldsymbol{e}_k, \omega \boldsymbol{a}\rangle - 2\pi \mathrm{i}\int_{I_k(\boldsymbol{a})} A\right), \tag{2.7.4}$$

which proves the above assertion. This equation also shows that the description of the holonomy is also independent of the choice of the representative $\boldsymbol{a}$ in $p(I_k(\boldsymbol{a}))$, i.e., if $p(I_k(\boldsymbol{a})) = p(I_k(\boldsymbol{a}'))$, the associated holonomies have the same value though they look different from each other at $\boldsymbol{a}$ and $\boldsymbol{a}'$. We now turn to the closed curve $C_\epsilon\big(\boldsymbol{c}_{\mathbf{0}}^{(j)}\big)$ with which we have defined the flux in (2.3.2). Let this curve be parameterized with $0 \leqslant t \leqslant 2\pi$. Then the associated holonomy is calculated as

$$[(x_0(2\pi), \boldsymbol{x}(2\pi))] = \left[\left(x_0(0) - \int_{C_\epsilon(\boldsymbol{c}_{\mathbf{0}}^{(j)})} A, \boldsymbol{x}(0)\right)\right] = [(x_0(0), \boldsymbol{x}(0))]\,\mathrm{e}^{-2\pi \mathrm{i}\rho_j(A)}. \tag{2.7.5}$$

Thus, we have proved the following.

**Proposition 2.13.** *For $\alpha_A \in \mathcal{C}\big(\dot{P}^3_\omega\big)$, the holonomies of the cycles $p(I_k(\boldsymbol{a})), k = 1, 2$, with respect to $[(a_0, \boldsymbol{a})] \in \dot{P}^3_\omega$ and those of the closed circles $C_\epsilon\big(\boldsymbol{c}_{\mathbf{0}}^{(j)}\big)$, $j = 1, \ldots, N$, are given by*

$$\exp(-2\pi \mathrm{i}\langle \boldsymbol{e}_k, \omega \boldsymbol{a}\rangle - 2\pi \mathrm{i} p_k(\boldsymbol{a}, A)), \qquad \mathrm{e}^{-2\pi \mathrm{i}\rho_j(A)}, \tag{2.7.6}$$

*respectively.*

**Remark.** Since $\langle \boldsymbol{e}_k, \omega \boldsymbol{a}\rangle, k = 1, 2$, are independent of the choice of connections, it turns out that the connections $\alpha_A$ and $\alpha_{A'}$ have the same holonomies associated with the cycle $p(I_k(\boldsymbol{a})), k = 1, 2$, and with the closed curves $C_\epsilon\big(\boldsymbol{c}_{\mathbf{0}}^{(j)}\big)$, $j = 1, \ldots, N$, if and only if the conditions (2.5.7) and (2.5.8) are satisfied.

## 3. Quantum systems

So far, we have studied A-B connections on $\dot{P}^3_\omega$. We now study A-B quantum systems coupled with the A-B connections. In accordance with the isomorphism of $\mathcal{C}_{\text{flat}}\big(\dot{P}^3_\omega\big)$ to $\mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$, the A-B quantum system to be defined on an associated complex line bundle over $\dot{T}^2$ will be unitarily brought into an A-B quantum system to be defined on a punctured plane $\dot{\mathbf{R}}^2$ along with wavefunctions satisfying a shift condition. The position operators and the momentum operators coupled with the A-B connection are defined and discussed accordingly. Further, A-B Hamiltonian operators are defined and studied with focus on the deficiency index. For the A-B Hamiltonian with all fluxes quantized, eigenvalues and eigenfunctions will be obtained.

### *3.1. Complex line bundles*

Let $\chi_n$ be a unitary irreducible representation of $U(1)$,

$$\chi_n(g) = g^n, \qquad g \in U(1), \qquad n \in \mathbf{Z}, \tag{3.1.1}$$

which defines an equivalent relation on $\dot{P}^3_\omega \times \mathbf{C}$ through

$$(u, z) \sim (u \cdot g, \chi_n(g^{-1})z), \qquad (u, z) \in \dot{P}^3_\omega \times \mathbf{C}. \tag{3.1.2}$$

This determines the vector bundle

$$\pi_{\omega,n} : \mathbf{E}_{\omega,n} = \big(\dot{P}^3_\omega \times \mathbf{C}\big)\big/\!\sim\; \longrightarrow \dot{T}^2, \tag{3.1.3}$$

which is called a complex line bundle associated with $\dot{P}^3_\omega$, of which the base space, the fibre and the structure group are $\dot{T}^2$, $\mathbf{C}$ and $U(1)$, respectively. We denote by $[(u,z)]_n \in \mathbf{E}_{\omega,n}$ the equivalence class of $(u,z)$. Further, by $\Gamma(\dot{T}^2, \mathbf{E}_{\omega,n})$ we denote the space of smooth sections in $\mathbf{E}_{\omega,n}$.

### *3.2. Sections and equivariant functions*

A smooth function $\psi : \dot{P}^3_\omega \to \mathbf{C}$ is said to be $\chi_n$ equivariant, if it satisfies that

$$\psi(u\cdot g) = \chi_n(g^{-1})\psi(u), \qquad u \in \dot{P}^3_\omega, \qquad g \in U(1). \tag{3.2.1}$$

Let us denote the set of $\chi_n$-equivariant functions on $\dot{P}^3_\omega$ by

$$\mathcal{E}_n(\dot{P}^3_\omega) = \{\psi : \dot{P}^3_\omega \to \mathbf{C} \mid R_g^*\psi = \chi_n(g^{-1})\psi, g \in U(1)\}. \tag{3.2.2}$$

It is well known that there is a one-to-one correspondence between $\mathcal{E}_n(\dot{P}^3_\omega)$ and $\Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$; to a given $\psi \in \mathcal{E}_n(\dot{P}^3_\omega)$, there corresponds a section $\sigma \in \Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$ through

$$\sigma(\pi_\omega(u)) = [(u, \psi(u))]_n, \qquad u \in \dot{P}^3_\omega. \tag{3.2.3}$$

We denote the isomorphism of $\mathcal{E}_n(\dot{P}^3_\omega)$ to $\Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$ by

$$\gamma : \mathcal{E}_n(\dot{P}^3_\omega) \longrightarrow \Gamma(\dot{T}^2; \mathbf{E}_{\omega,n}). \tag{3.2.4}$$

Furthermore, we define a function space $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ to be

$$C^\infty_{\omega,n}(\dot{\mathbf{R}}^2) = \{f \in C^\infty(\dot{\mathbf{R}}^2) | (T^*_{\boldsymbol{m}} f)(\boldsymbol{x}) = \mathrm{e}^{2\pi \mathrm{i} n\langle \boldsymbol{m}, \omega\boldsymbol{x}\rangle} f(\boldsymbol{x}), \boldsymbol{m} \in \mathbf{Z}^2\}. \tag{3.2.5}$$

We will refer to the property satisfied by $f \in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ as the shift property. With $f \in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$, we associate a function $\widetilde{\psi_f}$ on $\mathbf{R}\times\dot{\mathbf{R}}^2$ by

$$\widetilde{\psi_f}(x_0, \boldsymbol{x}) = \mathrm{e}^{-2\pi \mathrm{i} n x_0} f(\boldsymbol{x}). \tag{3.2.6}$$

$\widetilde{\psi_f}$ is invariant under the left action of $\mathbf{Z} \times {}_\omega \mathbf{Z}^2$, which is easily shown on account of the shift property of $f \in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$. Thus, $\widetilde{\psi_f}$ projects to a function $\psi_f$ on $\dot{P}^3_\omega$ through

$$\Pi^*_\omega \psi_f = \widetilde{\psi_f}. \tag{3.2.7}$$

As it is easily shown, $\psi_f$ is $\chi_n$ equivariant under the right $U(1)$ action. Thus, we obtain a vector space homomorphism

$$\widetilde{\gamma} : C^\infty_{\omega,n}(\dot{\mathbf{R}}^2) \longrightarrow \mathcal{E}_n(\dot{P}^3_\omega) : f \longmapsto \widetilde{\gamma} f = \psi_f. \tag{3.2.8}$$

**Lemma 3.1.** *$\widetilde{\gamma}$ is a vector space isomorphism of $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ to $\mathcal{E}_n(\dot{P}^3_\omega)$.*

**Proof.** Clearly, the homomorphism $\widetilde{\gamma}$ is injective. We show that $\widetilde{\gamma}$ is surjective. For a $\chi_n$-equivariant function $\psi \in \mathcal{E}_n(\dot{P}^3_\omega)$, we define a function $f$ on $\dot{\mathbf{R}}^2$ to be $f(\boldsymbol{x}) = (\Pi^*_\omega\psi)(0, \boldsymbol{x})$. Since $f$ transforms according to

$$\begin{aligned} f(\boldsymbol{x}+\boldsymbol{m}) &= (\Pi^*_\omega\psi)(0, \boldsymbol{x}+\boldsymbol{m}) = \psi([(0, \boldsymbol{x}+\boldsymbol{m})]) \\ &= \psi([(0, \boldsymbol{m})\cdot(-\langle \boldsymbol{m}, \omega\boldsymbol{x}\rangle, \boldsymbol{x})]) = \psi([(0, \boldsymbol{x})]\,\mathrm{e}^{-2\pi\mathrm{i}\langle \boldsymbol{m}, \omega\boldsymbol{x}\rangle}) \\ &= \mathrm{e}^{2\pi \mathrm{i} n\langle \boldsymbol{m}, \omega\boldsymbol{x}\rangle}\psi([(0, \boldsymbol{x})]) = \mathrm{e}^{2\pi \mathrm{i} n\langle \boldsymbol{m}, \omega\boldsymbol{x}\rangle} f(\boldsymbol{x}), \end{aligned} \tag{3.2.9}$$

$f$ proves to be an element of $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$. Further, we can verify that $f$ maps to $\psi$; for any $(\boldsymbol{x}_0, \boldsymbol{x}) \in \mathbf{R}\times\dot{\mathbf{R}}^2$,

$$\begin{aligned} \widetilde{\gamma} f([(x_0, \boldsymbol{x})]) &= \mathrm{e}^{-2\pi \mathrm{i} n x_0} f(\boldsymbol{x}) = \mathrm{e}^{-2\pi \mathrm{i} n x_0}(\Pi^*_\omega\psi)(0, \boldsymbol{x}) \\ &= \mathrm{e}^{-2\pi \mathrm{i} n x_0}\psi([(0, \boldsymbol{x})]) = \psi([(0, \boldsymbol{x})]\cdot \mathrm{e}^{2\pi\mathrm{i} x_0}) \\ &= \psi([(x_0, \boldsymbol{x})]). \end{aligned} \tag{3.2.10}$$

Thus, $\widetilde{\gamma}$ is surjective. This ends the proof. □

**Corollary 3.2.** *$\Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$, $\mathcal{E}_n(\dot{P}^3_\omega)$ and $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ are isomorphic to one another as complex vector spaces.*

Principal $U(1)$-bundles $\dot{P}^3_\omega$ are characterized by anti-symmetric parts of matrices $\omega$ (see proposition 2.1), so that vector space isomorphisms among $\mathcal{E}_n(\dot{P}^3_\omega)$, $\omega \in M(2, \mathbf{Z})$, are determined by the anti-symmetric part of $\omega$. Likewise, isomorphisms among $\Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$ and among $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ will be determined by the anti-symmetric parts of $\omega$.

**Proposition 3.3.** *Let $\tau \in M(2, \mathbf{Z})$ be a symmetric matrix. We assume that, for all $\boldsymbol{m} \in \mathbf{Z}^2$, $\langle \boldsymbol{m}, \tau\boldsymbol{m}\rangle$ are even integers. Then, there exists an isomorphism of $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ to $C^\infty_{\omega+\tau,n}(\dot{\mathbf{R}}^2)$. Further, for a diagonal matrix $\Delta \in M(2, \mathbf{Z})$, there exists an isomorphism of $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ to $C^\infty_{\omega+\Delta,n}(\dot{\mathbf{R}}^2)$.*

**Proof.** It is easy to show that the linear map,

$$C^\infty_{\omega,n}(\dot{\mathbf{R}}^2) \to C^\infty_{\omega+\tau,n}(\dot{\mathbf{R}}^2) : f \mapsto \mathrm{e}^{\pi i n\langle \boldsymbol{x}, \tau\boldsymbol{x}\rangle} f, \tag{3.2.11}$$

is an isomorphism. Furthermore, $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ is isomorphic to $C^\infty_{\omega+\Delta,n}(\dot{\mathbf{R}}^2)$ through the map

$$C^\infty_{\omega,n}(\dot{\mathbf{R}}^2) \to C^\infty_{\omega+\Delta,n}(\dot{\mathbf{R}}^2) : f \mapsto \mathrm{e}^{\pi i n(\langle \boldsymbol{x}, \Delta\boldsymbol{x}\rangle + \langle \boldsymbol{\delta}, \boldsymbol{x}\rangle)} f, \tag{3.2.12}$$

where $\boldsymbol{\delta} \in \mathbf{Z}^2$ is defined to be $\boldsymbol{\delta} = (d_1, d_2)^{\mathrm{T}}$ for $\Delta = \mathrm{diag}(d_1, d_2) \in M(2, \mathbf{Z})$. This completes the proof. □

**Corollary 3.4.** *Let $\omega, \omega' \in M(2, \mathbf{Z})$ and $n \in \mathbf{Z}$. If $\omega$ and $\omega'$ share the same anti-symmetric part, then $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ and $C^\infty_{\omega',n}(\dot{\mathbf{R}}^2)$ are isomorphic to each other. Consequently, $\Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$ and $\Gamma(\dot{T}^2; \mathbf{E}_{\omega',n})$ are isomorphic to each other.*

**Proof.** Since $\omega$ is decomposed as in (2.1.13), proposition 3.3 implies that $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ is isomorphic to $C^\infty_{\omega',n}(\dot{\mathbf{R}}^2)$, where $\omega' := \begin{pmatrix} 0 & \omega_{12}-\omega_{21} \\ 0 & 0 \end{pmatrix}$. This ends the proof. □

### *3.3. Covariant derivatives*

So far we have discussed three spaces $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$, $\mathcal{E}_n(\dot{P}^3_\omega)$ and $\Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$, which are isomorphic to one another. As is well known, $\Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$ is endowed with the covariant differentiation associated with the connection on $\dot{P}^3_\omega$. Operators corresponding to the covariant differentiation will be defined on $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$. In what follows, the space of smooth vector fields on a manifold $M$ will be denoted by $\mathcal{X}(M)$.

Let $\alpha_A$ be a connection form on the principal $U(1)$-bundle $\dot{P}^3_\omega$ and $X$ a vector field on $\dot{T}^2$. The horizontal lift $X^* \in \mathcal{X}(\dot{P}^3_\omega)$ of $X$ with respect to $\alpha_A$ is determined by the two conditions:

$$(\mathrm{HL1})(\pi_\omega)_* X^* = X \qquad \text{and} \qquad (\mathrm{HL2})\alpha_A(X^*) = 0. \tag{3.3.1}$$

For a vector field $X$ on $\dot{T}^2$ and a section $\sigma \in \Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$, the covariant derivative, $\nabla_X\sigma$, of $\sigma$ with respect to $X$ is defined as

$$\nabla_X\sigma = \gamma X^*(\gamma^{-1}\sigma). \tag{3.3.2}$$

In what follows, we define an operator $P_X$ on $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ associated with the covariant differentiation $\nabla_X$ on $\mathcal{E}_n(\dot{P}^3_\omega)$. To the vector field $X^*$ on the principal bundle $\dot{P}^3_\omega$, there corresponds a unique lift $X^\sharp$ on $\mathbf{R} \times \dot{\mathbf{R}}^2$ with $(\Pi_\omega)_* X^\sharp = X^*$, if $X \in \mathcal{X}(\dot{T}^2)$ is extended to be a vector field on $\dot{\mathbf{R}}^2$ by periodicity. We then define the operator $P_X$ on $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ by setting

$$(P_X f)(\boldsymbol{x}) = (X^\sharp \widetilde{\psi_f})(0, \boldsymbol{x}), \qquad f \in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2). \tag{3.3.3}$$

Since $(\Pi_\omega^*\alpha_A)(X^\sharp) = 0$, an integral curve $(x_0(t), \boldsymbol{x}(t))$ of $X^\sharp$ is a solution to the differential equation

$$\frac{\mathrm{d}x_0(t)}{\mathrm{d}t} + A_{\boldsymbol{x}(t)}(X) = 0 \qquad \text{and} \qquad \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = X_{\boldsymbol{x}(t)}, \tag{3.3.4}$$

where $X$ is viewed as a vector field on $\dot{\mathbf{R}}^2$. The curve $\Pi_\omega(x_0(t), \boldsymbol{x}(t))$ in $\dot{P}_\omega^3$ is an integral curve of $X^*$, which is a horizontal lift of the curve $p(\boldsymbol{x}(t))$ with the tangent $X \in \mathcal{X}(\dot{T}^2)$. Then, equation (3.3.3) is expressed as

$$\begin{aligned}
(P_X f)(\boldsymbol{x}) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{e}^{-2\pi\mathrm{i}nx_0(t)} f(\boldsymbol{x}(t)) \\
&= \left.\frac{\mathrm{d}f(\boldsymbol{x}(t))}{\mathrm{d}t}\right|_{t=0} - 2\pi\mathrm{i}n\dot{x}_0(0) f(\boldsymbol{x}) \\
&= (Xf + 2\pi\mathrm{i}nA(X)f)(\boldsymbol{x}),
\end{aligned} \tag{3.3.5}$$

where $(x_0(t), \boldsymbol{x}(t))$ is a solution to (3.3.4), passing $(0, \boldsymbol{x}) \in \mathbf{R} \times \dot{\mathbf{R}}^2$ at $t = 0$. In particular, for the vector field $\partial_k = \frac{\partial}{\partial x_k} \in \mathcal{X}(\dot{T}^2)$, $P_{\partial_k}$ is put in the form

$$P_{\partial_k} = \frac{\partial}{\partial x_k} + 2\pi\mathrm{i}nA_k, \qquad k = 1, 2, \tag{3.3.6}$$

which will be used to define momentum operators coupled with the A-B potential.

For $\boldsymbol{m} \in \mathbf{Z}^2$, $T_{\boldsymbol{m}}^*$ and $Xf$ are composed to give

$$\begin{aligned}
(T_{\boldsymbol{m}}^*(Xf))(\boldsymbol{x}) &= (Xf)(T_{\boldsymbol{m}}(\boldsymbol{x})) \\
&= (\mathrm{d}f)_{T_{\boldsymbol{m}}(\boldsymbol{x})}(X) = (\mathrm{d}f)_{T_{\boldsymbol{m}}(\boldsymbol{x})}(T_{\boldsymbol{m}*}X) \\
&= (X(T_{\boldsymbol{m}}^* f))(\boldsymbol{x}),
\end{aligned} \tag{3.3.7}$$

where use has been made of $T_{\boldsymbol{m}}^* X = X$ on $\dot{\mathbf{R}}^2$. Furthermore, $X(T_{\boldsymbol{m}}^* f)$ can be calculated as

$$\begin{aligned}
(X(T_{\boldsymbol{m}}^* f))(\boldsymbol{x}) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\boldsymbol{x}(t) + \boldsymbol{m}) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{e}^{2\pi\mathrm{i}n\langle \boldsymbol{m}, \omega\boldsymbol{x}(t)\rangle} f(\boldsymbol{x}(t)) \\
&= \mathrm{e}^{2\pi\mathrm{i}n\langle \boldsymbol{m}, \omega\boldsymbol{x}\rangle} (Xf(\boldsymbol{x}) + 2\pi\mathrm{i}n\langle \boldsymbol{m}, \omega\dot{\boldsymbol{x}}(0)\rangle f(\boldsymbol{x})) .
\end{aligned} \tag{3.3.8}$$

Then, from (3.3.5), (3.3.7) and (3.3.8), it follows that

$$\begin{aligned}
(T_{\boldsymbol{m}}^*(P_X f))(\boldsymbol{x}) &= (T_{\boldsymbol{m}}^* Xf)(\boldsymbol{x}) + 2\pi\mathrm{i}n(T_{\boldsymbol{m}}^* A(X))(\boldsymbol{x})(T_{\boldsymbol{m}}^* f)(\boldsymbol{x}) \\
&= \mathrm{e}^{2\pi\mathrm{i}n\langle \boldsymbol{m}, \omega\boldsymbol{x}\rangle} (Xf(\boldsymbol{x}) + 2\pi\mathrm{i}n\langle \boldsymbol{m}, \omega\dot{\boldsymbol{x}}(0)\rangle f(\boldsymbol{x})) \\
&\quad + 2\pi\mathrm{i}n(A(X)(\boldsymbol{x}) - \langle \boldsymbol{m}, \omega\dot{\boldsymbol{x}}(0)\rangle) \mathrm{e}^{2\pi\mathrm{i}n\langle \boldsymbol{m}, \omega\boldsymbol{x}\rangle} f(\boldsymbol{x}) \\
&= \mathrm{e}^{2\pi\mathrm{i}n\langle \boldsymbol{m}, \omega\boldsymbol{x}\rangle} (P_X f)(\boldsymbol{x}),
\end{aligned} \tag{3.3.9}$$

so that the operation with $P_X$ is closed in $C_{\omega,n}^\infty(\dot{\mathbf{R}}^2)$. We note here that, from (3.3.5), $P_X$ has the following properties: for $X, Y \in \mathcal{X}(\dot{T}^2)$, $h \in C^\infty(\dot{T}^2)$ and $f \in C_{\omega,n}^\infty(\dot{\mathbf{R}}^2)$,

$$\begin{aligned}
&P_{X+Y} f = P_X f + P_Y f, \\
&P_{hX} f = h P_X f, \\
&P_X(hf) = (Xh) f + h(P_X f),
\end{aligned} \tag{3.3.10}$$

where $X$ and $h$ are regarded as being extended on $\dot{\mathbf{R}}^2$ by periodicity. These properties are analogous to those of the covariant derivative of $\sigma \in \Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$.

We now show that $P_X$ is related to $X^*$ by $\widetilde{\gamma} : C_{\omega,n}^\infty(\dot{\mathbf{R}}^2) \to \mathcal{E}_n(\dot{P}_\omega^3)$. Using the fact that

$$X^\sharp \widetilde{\psi}_f(x_0, \boldsymbol{x}) = \mathrm{e}^{-2\pi\mathrm{i}nx_0} X^\sharp \widetilde{\psi}_f(0, \boldsymbol{x}) = \mathrm{e}^{-2\pi\mathrm{i}nx_0}(P_X f)(\boldsymbol{x}) \tag{3.3.11}$$

and that

$$X^\sharp\widetilde{\psi_f}(x_0,\boldsymbol{x}) = X^\sharp(\Pi_\omega^*\psi_f)(x_0,\boldsymbol{x}) = (\Pi_{\omega*}X^\sharp)\psi_f([(x_0,\boldsymbol{x})])$$
$$= X^*\psi_f([(x_0,\boldsymbol{x})]) = \Pi_\omega^*(X^*\psi_f)(x_0,\boldsymbol{x}), \tag{3.3.12}$$

we obtain

$$\widetilde{\psi_{P_X f}}(x_0,\boldsymbol{x}) = \mathrm{e}^{-2\pi\mathrm{i}nx_0}(P_X f)(\boldsymbol{x}) = (X^\sharp\widetilde{\psi_f})(x_0,\boldsymbol{x}) = (\Pi_\omega^*(X^*\psi_f))(x_0,\boldsymbol{x}). \tag{3.3.13}$$

On the other hand, since $P_X f \in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$, there is a $\chi_n$-equivariant function $\psi_{P_X f}$ such that $\widetilde{\psi_{P_X f}} = \Pi_\omega^*(\psi_{P_X f})$. Comparing this with (3.3.13) along with the definition of $\widetilde{\gamma}$, we have

$$X^*\psi_f = \psi_{P_X f} = \widetilde{\gamma}P_X(\widetilde{\gamma}^{-1}\psi_f). \tag{3.3.14}$$

Thus, (3.3.2) and (3.3.14) are put together to give the following proposition.

**Proposition 3.5.** *The following diagram commutes:*

$$\begin{array}{ccccc}
C^\infty_{\omega,n}(\dot{\mathbf{R}}^2) & \xrightarrow{\tilde{\gamma}} & \mathcal{E}_n(\dot{P}^3_\omega) & \xrightarrow{\gamma} & \Gamma(\dot{T}^2;\mathbf{E}_{\omega,n}) \\
\downarrow P_X & & \downarrow X^* & & \downarrow \nabla_X \\
C^\infty_{\omega,n}(\dot{\mathbf{R}}^2) & \xrightarrow{\tilde{\gamma}} & \mathcal{E}_n(\dot{P}^3_\omega) & \xrightarrow{\gamma} & \Gamma(\dot{T}^2;\mathbf{E}_{\omega,n}).
\end{array} \tag{3.3.15}$$

### *3.4. Hilbert spaces*

We now deal with inner products defined on the spaces $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$, $\mathcal{E}_n(\dot{P}^3_\omega)$ and $\Gamma(\dot{T}^2;\mathbf{E}_{\omega,n})$. First, $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ is endowed with a natural inner product; for $f,g\in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$, the inner product is given by $\langle f,g\rangle_{C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)} = \int_{I^2}\overline{f(\boldsymbol{x})}g(\boldsymbol{x})\,\mathrm{d}^2\boldsymbol{x}$, where $I=[0,1]$ and $\mathrm{d}^2\boldsymbol{x} = \mathrm{d}x_1\,\mathrm{d}x_2$. Note that the definition is in keeping with the shift property of $f,g\in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$. Through the completion with this inner product, $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ is made into a Hilbert space

$$L^2_{\omega,n}(\mathbf{R}^2) = \left\{ f:\mathbf{R}^2\to\mathbf{C} \,\middle|\, \begin{array}{l}(T^*_{\boldsymbol{m}}f)(\boldsymbol{x}) = \mathrm{e}^{2\pi n\mathrm{i}\langle\boldsymbol{m},\omega\boldsymbol{x}\rangle}f(\boldsymbol{x}),\ \boldsymbol{m}\in\mathbf{Z}^2 \\ \displaystyle\int_{I^2}|f(\boldsymbol{x})|^2\,\mathrm{d}^2\boldsymbol{x} < +\infty\end{array}\right\}, \tag{3.4.1}$$

with the inner product

$$\langle f,g\rangle_{L^2_{\omega,n}(\mathbf{R}^2)} = \int_{I^2}\overline{f(\boldsymbol{x})}g(\boldsymbol{x})\,\mathrm{d}^2\boldsymbol{x}. \tag{3.4.2}$$

We proceed to $\mathcal{E}_n(\dot{P}^3_\omega)$, on which an inner product is defined by $\langle\psi,\phi\rangle_{\mathcal{E}_n(\dot{P}^3_\omega)} = \int_{\dot{P}^3_\omega}\overline{\psi(u)}\phi(u)\,\mathrm{d}\mu(u)$, where $\mathrm{d}\mu$ is the volume element on $\dot{P}^3_\omega$ described as $\mathrm{d}\mu = \mathrm{d}x_0\,\mathrm{d}^2\boldsymbol{x}$ with local coordinates $(x_0,\boldsymbol{x})$. We denote the completion of the inner product space $\mathcal{E}_n(\dot{P}^3_\omega)$ by

$$L^2_n(P^3_\omega) = \left\{ \psi: P^3_\omega\to\mathbf{C} \,\middle|\, \begin{array}{l}R^*_g\psi = \chi_n(g^{-1})\psi,\ g\in U(1) \\ \displaystyle\int_{P^3_\omega}|\psi(u)|^2\,\mathrm{d}\mu < +\infty\end{array}\right\}, \tag{3.4.3}$$

along with

$$\langle\psi_1,\psi_2\rangle_{L^2_n(P^3_\omega)} = \int_{P^3_\omega}\overline{\psi_1(u)}\psi_2(u)\,\mathrm{d}\mu(u), \qquad \psi_1,\psi_2\in L^2_n(P^3_\omega). \tag{3.4.4}$$

We now show that $\widetilde{\gamma}: C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)\to\mathcal{E}_n(\dot{P}^3_\omega)$ is an isometry. For $\psi_1,\psi_2\in\mathcal{E}_n(\dot{P}^3_\omega)$ and $f_1,f_2\in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$ related by $\psi_k = \widetilde{\gamma}f_k$, $k=1,2$, one has $f_k(\boldsymbol{x}) = (\Pi^*_\omega\psi_k)(0,\boldsymbol{x})$ and

$(\Pi_\omega^*\psi_k)(x_0, \boldsymbol{x}) = \mathrm{e}^{-2\pi\mathrm{i}nx_0}(\Pi_\omega^*\psi_k)(0, \boldsymbol{x}), k = 1, 2$. Then, the inner product of $f_1$ and $f_2$ is expressed and calculated as

$$\begin{aligned}\langle f_1, f_2\rangle_{C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)} &= \int_{I^2} \overline{(\Pi_\omega^*\psi_1)(0, \boldsymbol{x})}(\Pi_\omega^*\psi_2)(0, \boldsymbol{x})\,\mathrm{d}^2\boldsymbol{x} \\ &= \int_{I\times I^2} \overline{(\Pi_\omega^*\psi_1)(x_0, \boldsymbol{x})}(\Pi_\omega^*\psi_2)(x_0, \boldsymbol{x})\,\mathrm{d}x_0\,\mathrm{d}^2\boldsymbol{x} \\ &= \int_{\dot{P}^3_\omega} \overline{\psi_1(u)}\psi_2(u)\,\mathrm{d}\mu(u) = \langle\psi_1, \psi_2\rangle_{\mathcal{E}_n(\dot{P}^3_\omega)}.\end{aligned} \tag{3.4.5}$$

By completion, we obtain the following.

**Proposition 3.6.** *The map $\widetilde{\gamma} : C^\infty_{\omega,n}(\dot{\mathbf{R}}^2) \to \mathcal{E}_n(\dot{P}^3_\omega)$ is extended to a unitary operator from $L^2_{\omega,n}(\mathbf{R}^2)$ to $L^2_n(P^3_\omega)$, which we denote by the same symbol $\widetilde{\gamma}$,*

$$\langle f_1, f_2\rangle_{L^2_{\omega,n}(\mathbf{R}^2)} = \langle\widetilde{\gamma} f_1, \widetilde{\gamma} f_2\rangle_{L^2_n(P^3_\omega)}. \tag{3.4.6}$$

$\Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$ is endowed with a natural inner product. By completion with this inner product, $\Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$ can be made into a Hilbert space,

$$L^2(T^2; \mathbf{E}_{\omega,n}) = \left\{\sigma : T^2 \to \mathbf{E}_{\omega,n} \,\middle|\, \begin{array}{l}\pi_{\omega,n}\circ\sigma = \mathrm{id}_{T^2} \\ \int_{T^2} (\sigma(t), \sigma(t))_t\,\mathrm{d}v < \infty\end{array}\right\}, \tag{3.4.7}$$

along with the inner product

$$\langle\sigma_1, \sigma_2\rangle_{L^2(T^2;\mathbf{E}_{\omega,n})} = \int_{T^2} (\sigma_1(t), \sigma_2(t))_t\,\mathrm{d}v(t), \tag{3.4.8}$$

where $(\sigma_1(t), \sigma_2(t))_t$ is the fibre metric on each fibre $\pi^{-1}_{\omega,n}(t) \cong \mathbf{C}, t \in \dot{T}^2$, and where $\mathrm{d}v = \mathrm{d}x_1\,\mathrm{d}x_2$ is the volume element on $\dot{T}^2 = \dot{\mathbf{R}}^2/\mathbf{Z}^2$.

The map $\gamma : \mathcal{E}_n(\dot{P}^3_\omega) \to \Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$ is an isometry, since for $\psi_k \in \mathcal{E}_n(\dot{P}^3_\omega)$ and for $\sigma_k = \gamma\psi_k \in \Gamma(\dot{T}^2; \mathbf{E}_{\omega,n}), k = 1, 2$, one has

$$\begin{aligned}\langle\sigma_1, \sigma_2\rangle_{\Gamma(\dot{T}^2;\mathbf{E}_{\omega,n})} &= \int_{T^2} (\sigma_1(t), \sigma_2(t))_t\,\mathrm{d}v(t) \\ &= \int_{T^2} \overline{\psi_1(u)}\psi_2(u)\,\mathrm{d}v(t) \quad \left(u \in \pi^{-1}_\omega(t)\right) \\ &= \int_{T^2} \left(\int_{\mathrm{U}(1)} \overline{\psi_1(u\cdot g)}\psi_2(u\cdot g)\frac{1}{2\pi\mathrm{i}}g^{-1}\,\mathrm{d}g\right)\mathrm{d}v(t) \quad \left(u \in \pi^{-1}_\omega(t)\right) \\ &= \int_{P^3_\omega} \overline{\psi_1(u)}\psi_2(u)\,\mathrm{d}\mu(u) \\ &= \langle\psi_1, \psi_2\rangle_{\mathcal{E}_n(\dot{P}^3_\omega)}.\end{aligned} \tag{3.4.9}$$

**Proposition 3.7.** *The map $\gamma : \mathcal{E}_n(\dot{P}^3_\omega) \to \Gamma(\dot{T}^2; \mathbf{E}_{\omega,n})$ can be extended to a unitary operator from $L^2_n(P^3_\omega)$ to $L^2(T^2; \mathbf{E}_{\omega,n})$, which is denoted by the same symbol $\gamma$,*

$$\langle\psi_1, \psi_2\rangle_{L^2_n(P^3_\omega)} = \langle\gamma\psi_1, \gamma\psi_2\rangle_{L^2(T^2;\mathbf{E}_{\omega,n})}. \tag{3.4.10}$$

**Corollary 3.8.** *The Hilbert spaces $L^2_{\omega,n}(\mathbf{R}^2)$, $L^2_n(P^3_\omega)$ and $L^2(T^2; \mathbf{E}_{\omega,n})$ are isomorphic to one another through the two unitary operators $\gamma$ and $\widetilde{\gamma}$:*

$$L^2_{\omega,n}(\mathbf{R}^2) \xrightarrow{\widetilde{\gamma}} L^2_n(P^3_\omega) \xrightarrow{\gamma} L^2(T^2; \mathbf{E}_{\omega,n}). \tag{3.4.11}$$

Each of these spaces is interpreted as the space of wavefunctions, on which an A-B quantum system will be defined.

*3.5. Quantum systems on a punctured 2-torus*

In this section, we study quantum systems on a punctured 2-torus, which are defined on each of $L^2(T^2; \mathbf{E}_{\omega,n})$, $L^2_n(P^3_\omega)$ and $L^2_{\omega,n}(\mathbf{R}^2)$. We will work with $L^2_{\omega,n}(\mathbf{R}^2)$, and occasionally refer to $L^2(T^2; \mathbf{E}_{\omega,n})$. A quantum system is usually defined by assigning position and momentum operators along with their commutation relations. However, the position coordinates $x_k$ are not adequate as position operators, because $x_k$ are not observable quantities on the torus. Appropriate alternatives to $x_k$ are operators $Q_k$ defined to be

$$(Q_k f)(\boldsymbol{x}) = \mathrm{e}^{2\pi \mathrm{i} x_k} f(\boldsymbol{x}), \qquad k = 1, 2, \tag{3.5.1}$$

which are unitary operators on $L^2_{\omega,n}(\mathbf{R}^2)$. In view of (3.3.6), the momentum operators coupled with the A-B connection should be

$$P_k = -\mathrm{i} P_{\partial_k} = -\mathrm{i}\frac{\partial}{\partial x_k} + 2\pi n A_k, \qquad k = 1, 2, \tag{3.5.2}$$

where $A \in \mathcal{A}_\omega(\dot{\mathbf{R}}^2)$. We now take tentatively the domain of $P_k$ as

$$D(P_k) = \left\{ f \in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2) \,\middle|\, \operatorname{supp} f \cap \Lambda = \emptyset \right\}, \qquad k = 1, 2. \tag{3.5.3}$$

$P_1$, $P_2$ are symmetric operators. A quantum system on $L^2_{\omega,n}(\mathbf{R}^2)$ is then defined to be a quintuplet $\left(L^2_{\omega,n}(\mathbf{R}^2), Q_1, Q_2, P_1, P_2\right)$. According to proposition 3.5 and corollary 3.8, to this quantum system, there corresponds a quantum system,

$$(L^2(T^2; \mathbf{E}_{\omega,n}), U_1, U_2, -\mathrm{i}\nabla_{\partial_1}, -\mathrm{i}\nabla_{\partial_2}), \tag{3.5.4}$$

where $U_k$ and $-\mathrm{i}\nabla_{\partial_k}$ are operators on $L^2(T^2; \mathbf{E}_{\omega,n})$ related to $Q_k$ and $P_k$ by

$$U_k = (\gamma \circ \widetilde{\gamma}) Q_k (\gamma \circ \widetilde{\gamma})^{-1}, \qquad -\mathrm{i}\nabla_{\partial_k} = (\gamma \circ \widetilde{\gamma}) P_k (\gamma \circ \widetilde{\gamma})^{-1}, \qquad k = 1, 2, \tag{3.5.5}$$

respectively. In the following, we work mainly with $\left(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P_k\right)$.

Let $\{V_k(t)\}_{t\in\mathbf{R}}$, $k = 1, 2$, be one-parameter unitary groups on $L^2_{\omega,n}(\mathbf{R}^2)$ defined by

$$(V_k(t) f)(\boldsymbol{x}) = \exp\left(2\pi \mathrm{i} n \int_{\boldsymbol{x}}^{\boldsymbol{x}+t\boldsymbol{e}_k} A\right) f(\boldsymbol{x} + t\boldsymbol{e}_k), \qquad \text{a.e. } \boldsymbol{x}, \tag{3.5.6}$$

where $\int_{\boldsymbol{x}}^{\boldsymbol{x}+t\boldsymbol{e}_k}$ denotes the integration along the line segment from $\boldsymbol{x}$ to $\boldsymbol{x} + t\boldsymbol{e}_k$, $k = 1, 2$. We have here to verify that $V_k(t) f$ satisfy the shift property:

$$\begin{aligned}
(V_k(t) f)(\boldsymbol{x} + \boldsymbol{m}) &= \exp\left(2\pi \mathrm{i} n \int_{\boldsymbol{x}+\boldsymbol{m}}^{\boldsymbol{x}+\boldsymbol{m}+t\boldsymbol{e}_k} A\right) f(\boldsymbol{x} + \boldsymbol{m} + t\boldsymbol{e}_k) \\
&= \exp\left(2\pi \mathrm{i} n \int_{\boldsymbol{x}}^{\boldsymbol{x}+t\boldsymbol{e}_k} T^*_{\boldsymbol{m}} A\right) \mathrm{e}^{2\pi \mathrm{i} n \langle \boldsymbol{m}, \omega(\boldsymbol{x}+t\boldsymbol{e}_k)\rangle} f(\boldsymbol{x} + t\boldsymbol{e}_k) \\
&= \exp\left(2\pi \mathrm{i} n \int_{\boldsymbol{x}}^{\boldsymbol{x}+t\boldsymbol{e}_k} A - \langle \boldsymbol{m}, \omega\, \mathrm{d}\boldsymbol{y}\rangle\right) \mathrm{e}^{2\pi \mathrm{i} n \langle \boldsymbol{m}, \omega(\boldsymbol{x}+t\boldsymbol{e}_k)\rangle} f(\boldsymbol{x} + t\boldsymbol{e}_k) \\
&= \mathrm{e}^{2\pi \mathrm{i} n \langle \boldsymbol{m}, \omega \boldsymbol{x}\rangle} \exp\left(2\pi \mathrm{i} n \int_{\boldsymbol{x}}^{\boldsymbol{x}+t\boldsymbol{e}_k} A\right) f(\boldsymbol{x} + t\boldsymbol{e}_k) \\
&= \mathrm{e}^{2\pi \mathrm{i} n \langle \boldsymbol{m}, \omega \boldsymbol{x}\rangle} (V_k(t) f)(\boldsymbol{x}).
\end{aligned} \tag{3.5.7}$$

When differentiated with respect to $t$ at $t = 0$, equation (3.5.6) provides, for $f \in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (V_k(t) f)(\boldsymbol{x}) = \mathrm{i}(P_k f)(\boldsymbol{x}), \qquad k = 1, 2, \tag{3.5.8}$$

which shows that $P_k$ are the infinitesimal generators of $V_k(t)$. Thus, the Stone theorem yields the following proposition:

**Proposition 3.9** (Arai [5, 6])**.** *The symmetric operators $P_1$, $P_2$ are essentially self-adjoint and generate the one-parameter unitary groups*

$$(\mathrm{e}^{\mathrm{i}tP_k}f)(\boldsymbol{x}) = \exp\left(2\pi\mathrm{i}n\int_{\boldsymbol{x}}^{\boldsymbol{x}+t\boldsymbol{e}_k} A\right) f(\boldsymbol{x}+t\boldsymbol{e}_k), \qquad k=1,2, \tag{3.5.9}$$

*where* $\int_{\boldsymbol{x}}^{\boldsymbol{x}+t\boldsymbol{e}_k}$ *denotes the integration along the line segment from* $\boldsymbol{x}$ *to* $\boldsymbol{x}+t\boldsymbol{e}_k$.

**Remark.** For $t=1$, equation (3.5.9) becomes

$$\begin{aligned}(\mathrm{e}^{\mathrm{i}P_k}f)(\boldsymbol{x}) &= \exp\left(2\pi\mathrm{i}n\int_{I_k(\boldsymbol{x})} A\right)\cdot \mathrm{e}^{2\pi\mathrm{i}n\langle \boldsymbol{e}_k,\omega\boldsymbol{x}\rangle} f(\boldsymbol{x})\\ &= \chi_n(\mathrm{e}^{2\pi\mathrm{i}(\langle \boldsymbol{e}_k,\omega\boldsymbol{x}\rangle + p_k(\boldsymbol{x},A))}) f(\boldsymbol{x}).\end{aligned} \tag{3.5.10}$$

This implies that $\mathrm{e}^{\mathrm{i}P_k}$, $k=1,2$, are unitary multiplication operators. The phase factor on the right-hand side of the above equation is the representation of the holonomy of the connection $\alpha_A$ along the loop $p(C_k(\boldsymbol{x}))$ (see also (2.7.6) and (3.7.5)).

The commutators between position and canonical momentum operators are formally given by

$$[P_k, Q_l] = 2\pi Q_l\delta_{kl}, \qquad k,l=1,2, \tag{3.5.11}$$

$$[Q_1, Q_2] = 0, \tag{3.5.12}$$

$$[P_1, P_2] = 2\pi\mathrm{i}n\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right). \tag{3.5.13}$$

If $A \in \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$, the commutator $[P_1, P_2]$ vanishes on $D(P_k)$ (see (3.5.3)). However, the unitary groups $\mathrm{e}^{\mathrm{i}tP_k}$, $k=1,2$, do not commute on the whole Hilbert space $L^2_{\omega,n}(\mathbf{R}^2)$.

**Proposition 3.10** (Arai [5, 6])**.** *For $t, s \in \mathbf{R}$, the commutator of* $\mathrm{e}^{\mathrm{i}tP_1}$ *and* $\mathrm{e}^{\mathrm{i}sP_2}$ *is given by*

$$\mathrm{e}^{\mathrm{i}tP_1}\,\mathrm{e}^{\mathrm{i}sP_2}\,\mathrm{e}^{-\mathrm{i}tP_1}\,\mathrm{e}^{-\mathrm{i}sP_2} = \chi_n(\mathrm{e}^{2\pi\mathrm{i}\Phi_{t,s}}), \tag{3.5.14}$$

*where $\Phi_{t,s}$ is defined to be the integral of $A$,*

$$\Phi_{t,s}(\boldsymbol{x}) = \oint_{\ell(\boldsymbol{x};t,s)} A, \qquad a.e.\ \boldsymbol{x}, \tag{3.5.15}$$

*along the rectangular path $\ell(\boldsymbol{x};t,s)$,*

$$\boldsymbol{x} \to \boldsymbol{x}+t\boldsymbol{e}_1 \to \boldsymbol{x}+t\boldsymbol{e}_1+s\boldsymbol{e}_2 \to \boldsymbol{x}+s\boldsymbol{e}_2 \to \boldsymbol{x}. \tag{3.5.16}$$

*If $A \in \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$, the function $\Phi_{t,s}$ is expressed as*

$$\Phi_{t,s}(\boldsymbol{x}) = \mathrm{sgn}(st)\sum_{c_m^{(j)}\in S(\boldsymbol{x};t,s)} \rho_j(A), \qquad a.e.\ \boldsymbol{x}, \tag{3.5.17}$$

*where $S(\boldsymbol{x};t,s)$ is the region in $\mathbf{R}^2$ enclosed by the loop $\ell(\boldsymbol{x};t,s)$.*

**Proof.** We can verify the statement by calculating $\mathrm{e}^{\mathrm{i}tP_1}\,\mathrm{e}^{\mathrm{i}sP_2}\,\mathrm{e}^{-\mathrm{i}tP_1}\,\mathrm{e}^{-\mathrm{i}sP_2}$ by means of (3.5.9). □

**Remark.** The function $\Phi_{t,s}$ is not defined on the whole punctured plane $\dot{\mathbf{R}}^2$, but on the dense subset $\{\boldsymbol{x}\in\dot{\mathbf{R}}^2 \mid \ell(\boldsymbol{x};t,s)\cap\Lambda = \emptyset\}$. The quantity $\chi_n(\mathrm{e}^{2\pi\mathrm{i}\Phi_{t,s}})$ is the holonomy with respect to the connection $\nabla$ along the closed path $p(\ell(\boldsymbol{x};t,s))$ in $\dot{T}^2$, which is seen from proposition 3.16.

Equations (3.5.14) and (3.5.17) are put together to result in the following.

**Corollary 3.11.** *Let $A \in \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$. Then, the unitary operators $\mathrm{e}^{\mathrm{i}tP_1}$ and $\mathrm{e}^{\mathrm{i}sP_2}$ commute, if and only if $n\rho_j(A)$, $j = 1, \ldots, N$, are all integers.*

Hence, if $n\rho_j(A) \notin \mathbf{Z}$ for some $j$, the operators $\mathrm{e}^{\mathrm{i}tP_k}$ give an example of the Nelson phenomenon, which is the motivation of studies on singular vector potentials by Reech [17] and Arai [5, 6].

### *3.6. Unitary equivalence*

Let $A, A' \in \mathcal{A}_\omega(\dot{\mathbf{R}}^2)$, and put

$$P_k = -\mathrm{i}\frac{\partial}{\partial x_k} + 2\pi n A_k, \qquad P'_k = -\mathrm{i}\frac{\partial}{\partial x_k} + 2\pi n A'_k, \qquad k = 1, 2. \tag{3.6.1}$$

Two quantum systems $\left(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P_k\right)$ and $\left(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P'_k\right)$ are said to be unitarily equivalent, if there is a unitary operator $U : L^2_{\omega,n}(\mathbf{R}^2) \to L^2_{\omega,n}(\mathbf{R}^2)$ such that

$$Q_k = U^{-1} Q_k U, \qquad P'_k = U^{-1} P_k U, \qquad k = 1, 2, \tag{3.6.2}$$

where the domain of $P_k$ should be extended suitably, since $P_k$ are essentially self-adjoint.

We assume now that two flat connections $\alpha_A$ and $\alpha_{A'}$ are gauge equivalent. Then, there is a function $h \in C^\infty(\dot{T}^2; U(1))$ such that

$$A' = A + \frac{1}{2\pi \mathrm{i}} h^{-1}\, \mathrm{d}h. \tag{3.6.3}$$

Associated with $h$, an unitary operator $U$ on $L^2_{\omega,n}(\mathbf{R}^2)$ is defined by

$$(Uf)(\boldsymbol{x}) = (h(\boldsymbol{x}))^n f(\boldsymbol{x}), \qquad f \in L^2_{\omega,n}(\mathbf{R}^2). \tag{3.6.4}$$

The unitary operator $U$ proves to intertwine the operators $\{Q_k, P_k\}$ and $\{Q_k, P'_k\}$, where $P_k$, $P'_k$, $k = 1, 2$ are given in (3.6.1). In fact, a straightforward calculation provides

$$\begin{aligned}
(UP_kU^{-1}f)(\boldsymbol{x}) &= (h(\boldsymbol{x}))^n \left(\left(-\mathrm{i}\frac{\partial}{\partial x_k} + 2\pi n A_k\right)(h^{-n} f)\right)(\boldsymbol{x}) \\
&= -\mathrm{i}\frac{\partial f}{\partial x_k}(\boldsymbol{x}) + 2\pi n \left(A_k(\boldsymbol{x}) + \frac{1}{2\pi \mathrm{i}} h(\boldsymbol{x})^{-1} \frac{\partial h}{\partial x_k}(\boldsymbol{x})\right) f(\boldsymbol{x}) \\
&= \left(\left(-\mathrm{i}\frac{\partial}{\partial x_k} + 2\pi n A'_k\right) f\right)(\boldsymbol{x}) \\
&= (P'_k f)(\boldsymbol{x}).
\end{aligned} \tag{3.6.5}$$

It is easy to see that $UQ_kU^{-1} = Q_k$, $k = 1, 2$. Thus, we have shown the following proposition.

**Proposition 3.12.** *If two flat connections $\alpha_A$ and $\alpha_{A'}$ are gauge equivalent, two quantum systems $\left(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P_k\right)$ and $\left(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P'_k\right)$ are unitarily equivalent.*

A question now arises as to what conditions are necessary and sufficient for the quantum systems $(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P_k)$ and $(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P'_k)$ to be unitarily equivalent.

**Lemma 3.13.** *Let $A, A' \in \mathcal{Z}(\dot{\mathbf{R}}^2)$ and set*

$$\Phi_{t,s}(\boldsymbol{x}) = \oint_{\ell(\boldsymbol{x};t,s)} A, \qquad \Phi'_{t,s}(\boldsymbol{x}) = \oint_{\ell(\boldsymbol{x};t,s)} A', \qquad \text{a.e. } \boldsymbol{x}, \tag{3.6.6}$$

*where $\ell(\boldsymbol{x}; t, s)$ is the rectangular path given in (3.5.16). If two quantum systems $\left(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P_k\right)$ and $\left(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P'_k\right)$ are unitarily equivalent, the condition*

$$\chi_n(\mathrm{e}^{2\pi\mathrm{i}\Phi_{t,s}(\boldsymbol{x})}) = \chi_n(\mathrm{e}^{2\pi\mathrm{i}\Phi'_{t,s}(\boldsymbol{x})}), \qquad \textit{a.e.} \quad \boldsymbol{x}, \tag{3.6.7}$$

*holds for any* $t, s \in \mathbf{R}$*, so that for* $j = 1, 2, \ldots, N$*,*

$$\chi_n(\mathrm{e}^{2\pi \mathrm{i}\rho_j(A)}) = \chi_n(\mathrm{e}^{2\pi \mathrm{i}\rho_j(A')}). \tag{3.6.8}$$

*Furthermore, there exists a point* $\boldsymbol{a} \in D$ *such that*

$$\chi_n(\mathrm{e}^{2\pi \mathrm{i}p_k(\boldsymbol{a},A)}) = \chi_n(\mathrm{e}^{2\pi \mathrm{i}p_k(\boldsymbol{a},A')}), \qquad k = 1, 2. \tag{3.6.9}$$

**Proof.** If $\left(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P_k\right)$ and $\left(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P'_k\right)$ are unitarily equivalent, there is a unitary operator $U$ such that

$$UQ_k = Q_kU, \qquad UP'_k = P_kU, \qquad k = 1, 2. \tag{3.6.10}$$

Exponentiated, the second of the above equations provides $\mathrm{e}^{\mathrm{i}tP'_k} = U^{-1}\,\mathrm{e}^{\mathrm{i}tP_k}U$, $k = 1, 2$. Then, proposition 3.10 implies that

$$\exp(2\pi \mathrm{i}n\Phi'_{t,s}(\boldsymbol{x})) = U^{-1}\exp(2\pi \mathrm{i}n\Phi_{t,s}(\boldsymbol{x}))U \tag{3.6.11}$$

for any $t, s \in \mathbf{R}$ and for a.e. $\boldsymbol{x} \in \mathbf{R}^2$. The function $\exp(2\pi \mathrm{i}n\Phi_{t,s}(\boldsymbol{x}))$ is a periodic function on $\dot{\mathbf{R}}^2$. In fact, for all $\boldsymbol{m} \in \mathbf{Z}^2$, one has

$$\begin{aligned}\Phi_{t,s}(\boldsymbol{x}+\boldsymbol{m}) &= \oint_{\ell(\boldsymbol{x}+\boldsymbol{m};t,s)} A = \oint_{\ell(\boldsymbol{x};t,s)} T^*_{\boldsymbol{m}}A \\ &= \oint_{\ell(\boldsymbol{x};t,s)} (A - \langle \boldsymbol{m}, \omega\,\mathrm{d}\boldsymbol{x}\rangle) = \Phi_{t,s}(\boldsymbol{x}).\end{aligned} \tag{3.6.12}$$

Hence, $\mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}(\boldsymbol{x})}$ can be expanded into a Fourier series in the sense of the $L^2$-norm,

$$\mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}} = \sum_{\boldsymbol{m}\in\mathbf{Z}^2} d_{\boldsymbol{m}}\,\mathrm{e}^{2\pi \mathrm{i}\langle \boldsymbol{m},\boldsymbol{x}\rangle}. \tag{3.6.13}$$

For $M > 0$, we put

$$A_M = \sum_{\substack{\boldsymbol{m}=(m_1,m_2)^{\mathrm{T}}\in\mathbf{Z}^2,\\ |m_1|+|m_2|\leqslant M}} d_{\boldsymbol{m}}\,\mathrm{e}^{2\pi \mathrm{i}\langle \boldsymbol{m},\boldsymbol{x}\rangle}. \tag{3.6.14}$$

Then, $A_M$ converge to $\mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}}$ in the sense of the $L^2$-norm as $M \to \infty$. Since $A_M$ can be described as a polynomial in $Q_1, Q_2$ with finite degree, and since $U$ and $Q_k$ commute, $U$ is commutative with $A_M$ for any $M > 0$. For an arbitrary $f \in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$, the norm $\|(\mathrm{e}^{2\pi \mathrm{i}n\Phi'_{t,s}} - \mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}})f\|$ is estimated as follows:

$$\begin{aligned}&\|(\mathrm{e}^{2\pi \mathrm{i}n\Phi'_{t,s}} - \mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}})f\|_{L^2_{\omega,n}(\mathbf{R}^2)} \\ &\quad = \|(U^{-1}\,\mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}}U - \mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}})f\|_{L^2_{\omega,n}(\mathbf{R}^2)} \\ &\quad \leqslant \|(U^{-1}\,\mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}}U - A_M)f\|_{L^2_{\omega,n}(\mathbf{R}^2)} + \|(A_M - \mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}})f\|_{L^2_{\omega,n}(\mathbf{R}^2)} \\ &\quad \leqslant \|(A_M - \mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}})Uf\|_{L^2_{\omega,n}(\mathbf{R}^2)} + \|(A_M - \mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}})f\|_{L^2_{\omega,n}(\mathbf{R}^2)} \\ &\quad \leqslant 2\left(\int_{I^2} |A_M - \mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}}|^2\,\mathrm{d}^2\boldsymbol{x}\right)^{\frac{1}{2}} \|f\|_{L^2_{\omega,n}(\mathbf{R}^2)}.\end{aligned} \tag{3.6.15}$$

The right-hand side vanishes, as $M \to \infty$. Since $f$ is arbitrary, this shows that the operators $\mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}}$ and $\mathrm{e}^{2\pi \mathrm{i}n\Phi'_{t,s}}$ are equal as multiplication operators. However, from definition (3.6.6), $\Phi_{t,s}$ and $\Phi'_{t,s}$ are step functions, and both of them change synchronously, so that $\mathrm{e}^{2\pi \mathrm{i}n\Phi_{t,s}}$ and $\mathrm{e}^{2\pi \mathrm{i}n\Phi'_{t,s}}$ are equal as functions on $\dot{\mathbf{R}}^2$ except the points where they are not defined. This proves (3.6.7).

From (3.5.10), the unitary operators $e^{iP_k}, e^{iP'_k}$ are both multiplication operators. The equations $e^{iP'_k} = U^{-1} e^{iP_k} U$, $k = 1, 2$, then turn out to be

$$e^{2\pi i n(\langle e_k, \omega x\rangle + \int_{I_k(x)} A')} = U^{-1} e^{2\pi i n(\langle e_k, \omega x\rangle + \int_{I_k(x)} A)} U, \qquad k = 1, 2. \tag{3.6.16}$$

Since, for all $m \in \mathbf{Z}^2$, one has

$$\begin{aligned}\langle e_k, \omega(x+m)\rangle + \int_{I_k(x+m)} A &= \langle e_k, \omega(x+m)\rangle + \int_{I_k(x)} T_m^* A \\ &= \langle e_k, \omega x\rangle - \langle m, \omega e_k\rangle + \langle e_k, \omega m\rangle + \int_{I_k(x)} A,\end{aligned} \tag{3.6.17}$$

and since $\langle m, \omega e_k\rangle + \langle e_k, \omega m\rangle \in \mathbf{Z}$, $\exp(2\pi i n(\langle e_k, \omega x\rangle + \int_{I_k(x)} A))$ is a periodic function. Hence, in a similar manner to that applied to (3.6.11), it follows from (3.6.16) that $\exp(2\pi i n(\langle e_k, \omega x\rangle + \int_{I_k(x)} A))$ is equal to $\exp(2\pi i n(\langle e_k, \omega x\rangle + \int_{I_k(x)} A'))$ almost everywhere in $\dot{\mathbf{R}}^2$. Then, there is a point $a \in D$ such that $e^{2\pi i n(\langle e_k, \omega a\rangle + p_k(a, A))} = e^{2\pi i n(\langle e_k, \omega a\rangle + p_k(a, A'))}$, as the set $D$ is dense in $\dot{\mathbf{R}}^2$. This ends the proof of (3.6.9). Thus, the proof of lemma 3.13 is completed. □

We are now in a position to prove a main theorem of this section.

**Theorem 3.14.** *Let $A, A' \in \mathcal{Z}(\dot{\mathbf{R}}^2)$, and let $Q_k$ and $P_k, P'_k$, $k = 1, 2$, be operators defined by (3.5.1) and (3.6.1), respectively. Two quantum systems $\left(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P_k\right)$ and $\left(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P'_k\right)$ are unitarily equivalent, if and only if the following conditions are satisfied:*

$$\chi_n(e^{2\pi i \rho_j(A)}) = \chi_n(e^{2\pi i \rho_j(A')}), \qquad j = 1, 2, \ldots, N, \tag{3.6.18}$$

*and*

$$\chi_n(e^{2\pi i p_k(a, A)}) = \chi_n(e^{2\pi i p_k(a, A')}), \qquad k = 1, 2, \tag{3.6.19}$$

*where $a \in D$.*

**Proof.** Lemma 3.13 shows that equations (3.6.18) and (3.6.19) are necessary. We now show that equations (3.6.18) and (3.6.19) are sufficient. From (3.6.18) together with $dA = dA' = 0$ on $\dot{\mathbf{R}}^2$, it follows that for any $x \in \dot{\mathbf{R}}^2$, the quantity $\exp\left(2\pi i n \int_x^{x+e_k} (A' - A)\right)$ is defined independently of the choice of paths joining $x$ to $x + e_k$, so that this quantity determines a function on $\dot{\mathbf{R}}^2$. Furthermore, since $T^*_{e_k}(A' - A) = A' - A$, the differential of $\exp\left(2\pi i n \int_x^{x+e_k} (A' - A)\right)$ turns out to vanish, like (2.5.12). This and equation (3.6.19) are put together to imply that $\exp\left(2\pi i n \int_x^{x+e_k} (A' - A)\right) = 1$ for any $x \in \dot{\mathbf{R}}^2$.

We here define a function $h : \dot{\mathbf{R}}^2 \to U(1)$ to be

$$h(x) = \exp\left(2\pi i n \int_a^x (A' - A)\right), \tag{3.6.20}$$

where $a \in \dot{\mathbf{R}}^2$ is an arbitrarily chosen point. Like (2.5.14), this $h$ is a function on $\dot{\mathbf{R}}^2$. In the same manner as in (2.5.15), we can verify that $h$ is a periodic function on $\dot{\mathbf{R}}^2$ as well. Hence, we can define a unitary operator $U$ on $L^2_{\omega,n}(\mathbf{R}^2)$ to be

$$(Uf)(x) = h(x) f(x). \tag{3.6.21}$$

This unitary operator $U$ proves to intertwine the momentum operators $P_k$ and $P'_k$, $k = 1, 2$. In fact, a straightforward computation results in

$$\frac{1}{2\pi i n} h^{-1} \frac{\partial h}{\partial x_k} = A'_k - A_k, \qquad k = 1, 2, \tag{3.6.22}$$

and thereby one obtains, for any $f \in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$,

$$U^{-1}P_kUf = h^{-1}\left(-\mathrm{i}\frac{\partial}{\partial x_k} + 2\pi n A_k\right)hf = -\mathrm{i}\frac{\partial f}{\partial x_k} + \left(2\pi n A_k - \mathrm{i}h^{-1}\frac{\partial h}{\partial x_k}\right)f = P'_k f, \qquad k = 1, 2. \tag{3.6.23}$$

This ends the proof. □

Theorem 3.14 can be put in terms of quantum systems defined on $L^2(T^2; \mathbf{E}_{\omega,n})$.

**Theorem 3.15.** *Let $\alpha_A$ and $\alpha_{A'}$ be flat connections on $\dot{P}^3_\omega$, and $\nabla$ and $\nabla'$ the connections on $\mathbf{E}_{\omega,n}$ associated with $\alpha_A$ and $\alpha_{A'}$. The quantum systems $(L^2(T^2; \mathbf{E}_{\omega,n}), U_k, -\mathrm{i}\nabla_{\partial_k})$ and $(L^2(T^2; \mathbf{E}_{\omega,n}), U_k, -\mathrm{i}\nabla'_{\partial_k})$ are unitarily equivalent, if and only if equations (3.76) and (3.77) hold true.*

### *3.7. Holonomy in quantum systems*

In this section, we remark that conditions (3.6.18) and (3.6.19) in theorem 3.14 mean that the connections on $\mathbf{E}_{\omega,n}$ associated with $\alpha_A$ and with $\alpha_{A'}$ have the same holonomies.

Let $c(t)$ be a curve in $\dot{T}^2$. A section $\sigma$ defined along the curve $c(t)$ is said to be parallel along $c$, if $\nabla_{\dot{c}(t)}\sigma = 0$, where $\dot{c}(t)$ denotes the tangent vector to the curve $c$ at $c(t)$. The parallelism can be translated into $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$. Let $C : \boldsymbol{x} = \boldsymbol{x}(t), 0 \leqslant t \leqslant L$, be a curve in $\dot{\mathbf{R}}^2$. A function $F(\boldsymbol{x}(t))$ defined along the curve $C$ is called parallel along $C$, if it satisfies

$$P_{\dot{\boldsymbol{x}}(t)}F = -\mathrm{i}\frac{\mathrm{d}F}{\mathrm{d}t} + 2n\pi\mathrm{i}A\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)F = 0. \tag{3.7.1}$$

This is easily integrated to give

$$F(\boldsymbol{x}(t)) = \exp\left(-2n\pi\mathrm{i}\int_0^t A\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)\mathrm{d}\tau\right)F(\boldsymbol{x}_0). \tag{3.7.2}$$

We here set $\boldsymbol{x}(t) = \boldsymbol{a} + t\boldsymbol{e}_k, 0 \leqslant t \leqslant 1$, and assume that $F(\boldsymbol{a}) = f(\boldsymbol{a})$ with $f \in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$. As $f$ has the shift property, one has $f(\boldsymbol{a} + \boldsymbol{e}_k) = \mathrm{e}^{2n\pi\mathrm{i}\langle \boldsymbol{e}_k, \omega\boldsymbol{a}\rangle} f(\boldsymbol{a})$. Thus we obtain, at $t = 1$,

$$F(\boldsymbol{a} + \boldsymbol{e}_k) = \exp\left(-2n\pi\mathrm{i}\int_{I_k(\boldsymbol{a})} A\right)\mathrm{e}^{-2n\pi\mathrm{i}\langle \boldsymbol{e}_k, \omega\boldsymbol{a}\rangle} f(\boldsymbol{a} + \boldsymbol{e}_k), \tag{3.7.3}$$

which provides the holonomy associated with the closed curve $p(I_k(\boldsymbol{a}))$ on $\dot{T}^2$,

$$\exp(-2n\pi\mathrm{i}(p_k(\boldsymbol{a}, A) + \langle \boldsymbol{e}_k, \omega\boldsymbol{a}\rangle)). \tag{3.7.4}$$

For $C = C_\epsilon\big(\boldsymbol{c}_{\boldsymbol{0}}^{(j)}\big)$, the associated holonomy is given by $\exp(-2\pi\mathrm{i}n\rho_j(A))$, as is easily seen from (3.7.2). Thus, proposition 2.13 is represented as

**Proposition 3.16.** *For the connection on $\mathbf{E}_{\omega,n}$ associated with $\alpha_A \in \mathcal{C}\big(\dot{P}^3_\omega\big)$, the holonomies of the cycles $p(I_k(\boldsymbol{a})), k = 1, 2$, and those of the closed circles $C_\epsilon\big(\boldsymbol{c}_{\boldsymbol{0}}^{(j)}\big), j = 1, \ldots, N$, are given by*

$$\exp(-2\pi\mathrm{i}n\langle \boldsymbol{e}_k, \omega\boldsymbol{a}\rangle - 2\pi\mathrm{i}np_k(\boldsymbol{a}, A)), \qquad \mathrm{e}^{-2\pi\mathrm{i}n\rho_j(A)}, \tag{3.7.5}$$

*respectively.*

Theorem 3.14 is now restated as follows: two quantum systems $\big(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P_k\big)$ and $\big(L^2_{\omega,n}(\mathbf{R}^2), Q_k, P'_k\big)$ are unitarily equivalent, if and only if the holonomies of the cycles $p(I_k(\boldsymbol{a})), k = 1, 2$, and those of the closed circles $C_\epsilon\big(\boldsymbol{c}_{\boldsymbol{0}}^{(j)}\big), j = 1, \ldots, N$, in respective quantum systems coincide.

### *3.8. Aharonov–Bohm Hamiltonians*

In this and next sections, we assume that the matrix $\omega$ is anti-symmetric. From theorem 2.11, any flat connection $A \in \mathcal{Z}_\omega(\dot{\mathbf{R}}^2)$ with $\omega$ anti-symmetric is put in the form

$$A = \frac{1}{2\pi} \sum_{j=1}^{N} \nu_j \,\mathrm{Im}\big(\zeta\big(z - c_0^{(j)}\big)\,\mathrm{d}z\big) + \langle \varepsilon, \mathrm{d}\boldsymbol{x} \rangle, \tag{3.8.1}$$

up to gauge transformations, where $\nu_j$, $j = 1, \ldots, N$, are subject to the quantization condition $\sum_{j=1}^{N} \nu_j = \omega_{21} - \omega_{12}$ given in (2.3.3).

Since the momentum operators coupled with $A$ are given by (3.5.2), the A-B Hamiltonian is defined to be

$$H = \frac{1}{2}\big(P_1^2 + P_2^2\big) = \frac{1}{2} \sum_{k=1,2} \left(-\mathrm{i}\frac{\partial}{\partial x_k} + 2\pi n A_k\right)^2. \tag{3.8.2}$$

We take the domain of $H$ as

$$D(H) = \big\{ f \in C^\infty_{\omega,n}(\dot{\mathbf{R}}^2) \,\big|\, \mathrm{supp} f \cap \Lambda = \emptyset \big\}. \tag{3.8.3}$$

We now look into whether $H$ has a unique self-adjoint extension or not. Let $H^*$ and $\overline{H}$ denote the adjoint operator and the closure of $H$, respectively. Then the theory of symmetric operators [19] tells us that

$$D(H^*) = D(\overline{H}) \oplus \mathcal{K}_+ \oplus \mathcal{K}_-, \tag{3.8.4}$$

where $\mathcal{K}_\pm = \ker(H^* \mp \mathrm{i}I)$ are the deficiency subspaces of $H$. The Hamiltonian $H$ has a self-adjoint extension, if and only if both of the deficiency indices coincide, i.e.,

$$\dim \mathcal{K}_+ = \dim \mathcal{K}_-. \tag{3.8.5}$$

Moreover, the essential self-adjointness of $H$ is equivalent to $\dim \mathcal{K}_\pm = 0$. If the deficiency indices coincide, there is a one-to-one correspondence between self-adjoint extensions of $H$ and unitary transformations from $\mathcal{K}_+$ to $\mathcal{K}_-$. For a unitary operator $U : \mathcal{K}_+ \to \mathcal{K}_-$, the corresponding self-adjoint extension $H_U$ has the domain

$$D(H_U) = \big\{ f + f_+ + U f_+ \in L^2_{\omega,n}(\mathbf{R}^2) \,\big|\, f \in D(\overline{H}),\ f_+ \in \mathcal{K}_+ \big\} \tag{3.8.6}$$

and acts on $D(H_U)$ in the manner

$$H_U(f + f_+ + U f_+) = \overline{H} f + \mathrm{i} f_+ - \mathrm{i} U f_+. \tag{3.8.7}$$

Since the A-B Hamiltonian $H$ is semi-bounded below, both the deficiency indices coincide [19], so that $H$ has a self-adjoint extension. To get an explicit idea of the deficiency subspaces, we have to solve the equation $H^* f = \pm \mathrm{i} f$. However, without solving the equations, we can compute the deficiency indices for the A-B Hamiltonian $H$. From (3.8.4), we have

$$\dim \mathcal{K}_\pm = \tfrac{1}{2} \dim D(H^*)/D(\overline{H}). \tag{3.8.8}$$

In order to compute the dimension of $D(H^*)/D(\overline{H})$, we apply the localization principle [8, 13, 14]; each singularity should separately yield a contribution to the total deficiency index of $H$. Let $O$ be an open neighbourhood of $\Lambda = \big\{ \boldsymbol{c}_{\boldsymbol{m}}^{(j)} \big\}$ defined to be

$$O = \bigcup_{j=1}^{N} \bigcup_{\boldsymbol{m} \in \mathbf{Z}^2} \big\{ \boldsymbol{x} \in \mathbf{R}^2 \,\big|\, \big|\boldsymbol{x} - \boldsymbol{c}_{\boldsymbol{m}}^{(j)}\big| < \epsilon \big\}, \tag{3.8.9}$$

where $\epsilon$ is taken so small that $\big\{ \boldsymbol{x} \in \mathbf{R}^2 \,\big|\, \big|\boldsymbol{x} - \boldsymbol{c}_{\boldsymbol{m}}^{(j)}\big| < \epsilon \big\}$ may be disjoint to one another. We denote by $H_O$ the restriction of $H$ to

$$D(H_O) = \{ f \in D(H) \mid \mathrm{supp} f \subset O \}. \tag{3.8.10}$$

The operators $H_O^*$ and $\overline{H_O}$ are defined accordingly. We put

$$\mathcal{D}(H_O^*) = \{f \in D(H_O^*) \mid \operatorname{supp} f \subset O\}, \tag{3.8.11}$$

$$\mathcal{D}(\overline{H_O}) = \{f \in D(\overline{H_O}) \mid \operatorname{supp} f \subset O\} \subset \mathcal{D}(H_O^*). \tag{3.8.12}$$

A natural inclusion $\mathcal{D}(H_O^*) \hookrightarrow D(H^*)$ with $\mathcal{D}(\overline{H_O}) \hookrightarrow D(\overline{H})$ induces a well-defined linear map

$$\iota : \mathcal{D}(H_O^*)/\mathcal{D}(\overline{H_O}) \longrightarrow D(H^*)/D(\overline{H}). \tag{3.8.13}$$

**Lemma 3.17.** *The map $\iota$ gives an isomorphism of $\mathcal{D}(H_O^*)/\mathcal{D}(\overline{H_O})$ to $D(H^*)/D(\overline{H})$.*

(For the proof, see the appendix.)

Owing to this lemma together with the well-known fact that the deficiency indices of the usual A-B Hamiltonian are (1, 1) or (2, 2) according to whether the flux is an integer or not [1, 9], we can determine the deficiency indices of our A-B Hamiltonian.

**Theorem 3.18.** *Let $M := \#\{j \mid n\nu_j \notin \mathbf{Z}\}$ be the number of non-quantized fluxes. Then, the deficiency indices of the A-B Hamiltonian $H$ are given by $(M + N, M + N)$, i.e.,*

$$\dim \mathcal{K}_\pm = M + N. \tag{3.8.14}$$

*Hence, the family of self-adjoint extensions of $H$ is parameterized by the unitary group $U(M + N)$.*

**Proof.** A function $f$ in the domain $D(H_O)$ is naturally decomposed into

$$f(\boldsymbol{x}) = \sum_{j=1}^{N} \sum_{\boldsymbol{m}\in\mathbf{Z}^2} f_{\boldsymbol{m}}^{(j)}(\boldsymbol{x}), \tag{3.8.15}$$

where each $f_{\boldsymbol{m}}^{(j)}$ is a smooth function whose support is contained in the open ball $B_\epsilon\big(\boldsymbol{c}_{\boldsymbol{m}}^{(j)}\big) = \big\{\boldsymbol{x} \in \mathbf{R}^2 \,\big|\, \big|\boldsymbol{x} - \boldsymbol{c}_{\boldsymbol{m}}^{(j)}\big| < \epsilon\big\}$. Owing to the shift property of $f$, the functions $f_{\boldsymbol{m}}^{(j)}$ are related to one another by

$$\big(T_{\boldsymbol{m}-\boldsymbol{l}}^* f_{\boldsymbol{m}}^{(j)}\big)(\boldsymbol{x}) = \mathrm{e}^{2\pi\mathrm{i}n\langle \boldsymbol{m}-\boldsymbol{l},\omega\boldsymbol{x}\rangle} f_{\boldsymbol{l}}^{(j)}(\boldsymbol{x}), \qquad \boldsymbol{l}, \boldsymbol{m} \in \mathbf{Z}^2, \quad j = 1, \ldots, N. \tag{3.8.16}$$

This implies that $f$ is put in the form

$$f(\boldsymbol{x}) = \sum_{j=1}^{N} f^{(j)}(\boldsymbol{x}), \qquad f^{(j)}(\boldsymbol{x}) := \sum_{\boldsymbol{m}\in\mathbf{Z}^2} \mathrm{e}^{2\pi\mathrm{i}n\langle \boldsymbol{m},\omega\boldsymbol{x}\rangle} \big(T_{-\boldsymbol{m}}^* f_{\boldsymbol{0}}^{(j)}\big)(\boldsymbol{x}). \tag{3.8.17}$$

We here denote by $O_j$ the open sets

$$O_j = \bigcup_{\boldsymbol{m}\in\mathbf{Z}^2} \big\{\boldsymbol{x} \in \mathbf{R}^2 \big| \big|\boldsymbol{x} - \boldsymbol{c}_{\boldsymbol{m}}^{(j)}\big| < \epsilon\big\}, \qquad j = 1, \ldots, N. \tag{3.8.18}$$

Then, equation (3.8.17) implies that

$$D(H_O) = \bigoplus_{j=1}^{N} \{f^{(j)} \in D(H) \mid \operatorname{supp} f^{(j)} \subset O_j\} \tag{3.8.19}$$

$$\cong \bigoplus_{j=1}^{N} \big\{f_{\boldsymbol{0}}^{(j)} \in C_0^\infty(\mathbf{R}^2\backslash\Lambda) \,\big|\, \operatorname{supp} f_{\boldsymbol{0}}^{(j)} \subset B_\epsilon\big(\boldsymbol{c}_{\boldsymbol{0}}^{(j)}\big)\big\}, \tag{3.8.20}$$

where we have used the fact that our Hilbert space is endowed with the inner product (3.4.2) and that $f^{(j)}(\boldsymbol{x}) = \sum_{\boldsymbol{m}\in\mathbf{Z}^2} \mathrm{e}^{2\pi\mathrm{i}n\langle \boldsymbol{m},\omega\boldsymbol{x}\rangle}\big(T^*_{-\boldsymbol{m}} f^{(j)}_{\boldsymbol{0}}\big)(\boldsymbol{x})$. Here, we denote by $W_j$, $j = 1,\ldots,N$, the isomorphisms

$$W_j : \{f^{(j)} \in D(H) \mid \operatorname{supp} f^{(j)} \subset O_j\} \to \big\{ f^{(j)}_{\boldsymbol{0}} \in C_0^\infty(\mathbf{R}^2\backslash\Lambda) \,\big|\, \operatorname{supp} f^{(j)}_{\boldsymbol{0}} \subset B_\epsilon\big(\boldsymbol{c}^{(j)}_{\boldsymbol{0}}\big)\big\}. \quad (3.8.21)$$

Further, from (3.8.1) together with the properties of the Weierstrass zeta function (see lemma 2.4), there exists a smooth function $h_j$ on $B_\epsilon\big(\boldsymbol{c}^{(j)}_{\boldsymbol{0}}\big)$ such that

$$A\big|_{B_\epsilon(\boldsymbol{c}^{(j)}_{\boldsymbol{0}})\backslash\{\boldsymbol{c}^{(j)}_{\boldsymbol{0}}\}} = \frac{\nu_j}{2\pi}\left(\frac{-\big\langle e_2, \boldsymbol{x}-\boldsymbol{c}^{(j)}_{\boldsymbol{0}}\big\rangle \mathrm{d}x_1 + \big\langle e_1, \boldsymbol{x}-\boldsymbol{c}^{(j)}_{\boldsymbol{0}}\big\rangle \mathrm{d}x_2}{\big|\boldsymbol{x}-\boldsymbol{c}^{(j)}_{\boldsymbol{0}}\big|^2}\right) + \mathrm{d}h_j. \quad (3.8.22)$$

We note here that

$$T^*_{\boldsymbol{c}^{(j)}_{\boldsymbol{0}}}\left(\frac{-\big\langle e_2, \boldsymbol{x}-\boldsymbol{c}^{(j)}_{\boldsymbol{0}}\big\rangle \mathrm{d}x_1 + \big\langle e_1, \boldsymbol{x}-\boldsymbol{c}^{(j)}_{\boldsymbol{0}}\big\rangle \mathrm{d}x_2}{\big|\boldsymbol{x}-\boldsymbol{c}^{(j)}_{\boldsymbol{0}}\big|^2}\right) = \frac{-x_2\,\mathrm{d}x_1 + x_1\,\mathrm{d}x_2}{x_1^2+x_2^2}$$

is the usual Aharonov–Bohm gauge potential on $\mathbf{R}^2\backslash\{\boldsymbol{0}\}$. This observation means that by local gauge transformations $\mathrm{e}^{-2\pi\mathrm{i}nh_j}$ and translations $\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{c}^{(j)}_{\boldsymbol{0}}$, $H_O$ is locally expressed as the usual Aharonov–Bohm Hamiltonian on $\mathbf{R}^2$,

$$\begin{aligned} H^{n\nu_j}_{\mathrm{AB}} &= \frac{1}{2}\left\{\left(-\mathrm{i}\frac{\partial}{\partial x_1} - n\nu_j\frac{x_2}{x_1^2+x_2^2}\right)^2 + \left(-\mathrm{i}\frac{\partial}{\partial x_2} + n\nu_j\frac{x_1}{x_1^2+x_2^2}\right)^2\right\}, \\ D\big(H^{n\nu_j}_{\mathrm{AB}}\big) &= \big\{f \in C_0^\infty(\mathbf{R}^2\backslash\{0\}) \,\big|\, \operatorname{supp} f \subset B_\epsilon(\boldsymbol{0})\big\}. \end{aligned} \quad (3.8.23)$$

Thus, according to (3.8.20), $H_O$ is decomposed into

$$\begin{aligned} H_O &= \bigoplus_{j=1}^{N} W_j^{-1}\,\mathrm{e}^{-2\pi\mathrm{i}nh_j}\left(T^*_{-\boldsymbol{c}^{(j)}_{\boldsymbol{0}}} \circ H^{n\nu_j}_{\mathrm{AB}} \circ T^*_{\boldsymbol{c}^{(j)}_{\boldsymbol{0}}}\right)\mathrm{e}^{2\pi\mathrm{i}nh_j}\,W_j, \\ D(H_O) &= \bigoplus_{j=1}^{N} W_j^{-1}\,\mathrm{e}^{-2\pi\mathrm{i}nh_j}\,T^*_{-\boldsymbol{c}^{(j)}_{\boldsymbol{0}}}\left(D\big(H^{n\nu_j}_{\mathrm{AB}}\big)\right). \end{aligned} \quad (3.8.24)$$

As in lemma 3.17, the decomposition (3.8.24) gives rise to the isomorphism

$$\mathcal{D}(H_O^*)/\mathcal{D}(\overline{H_O}) \cong \bigoplus_{j=1}^{N} D\big(\big(H^{n\nu_j}_{\mathrm{AB}}\big)^*\big)\big/D\big(\overline{H^{n\nu_j}_{\mathrm{AB}}}\big). \quad (3.8.25)$$

We have here to make a comment on $D\big(H^{n\nu_j}_{\mathrm{AB}}\big)$. This domain defined above seems to be rather restrictive than the usual one $C_0^\infty(\mathbf{R}^2\backslash\{0\})$. However, it suffices for our purpose on account of the localization principle. In fact, in [7], the A-B Hamiltonian is studied on a disc. According to [1, 7, 9], the quotient space $D\big(\big(H^{n\nu_j}_{\mathrm{AB}}\big)^*\big)\big/D\big(\overline{H^{n\nu_j}_{\mathrm{AB}}}\big)$ is of dimension 2 or 4, depending on whether the quantity $n\nu_j$ is an integer or not. Then, equations (3.8.4), (3.8.8) and (3.8.25) are put together to conclude that the deficiency indices of $H$ are $(2M+N-M, 2M+N-M) = (M+N, M+N)$, where $M$ denotes the number of non-quantized fluxes. This ends the proof. □

### 3.9. *Eigenvalue problem for the A-B Hamiltonian*

We consider the eigenvalue problem, $Hf = Ef$, $f \in L^2_{\omega,n}(\mathbf{R}^2)$, of the A-B Hamiltonian (3.8.2). Since $P_1$ and $P_2$ are commutative on $C^\infty_{\omega,n}(\dot{\mathbf{R}}^2)$, the Hamiltonian $H$ is expressed as $H = \frac{1}{2}(P_1 + \mathrm{i}P_2)(P_1 - \mathrm{i}P_2)$. $P_1 + \mathrm{i}P_2$ and $P_1 - \mathrm{i}P_2$ are rewritten in the complex coordinates $z = x_1 + \mathrm{i}x_2$ and $\overline{z} = x_1 - \mathrm{i}x_2$ as

$$P_1 - \mathrm{i}P_2 = -2\mathrm{i}\frac{\partial}{\partial z} - \mathrm{i}\sum_{j=1}^{N} n\nu_j \zeta\left(z - c_0^{(j)}\right) + 2\pi n\overline{\varepsilon}, \tag{3.9.1a}$$

$$P_1 + \mathrm{i}P_2 = -2\mathrm{i}\frac{\partial}{\partial \overline{z}} + \mathrm{i}\sum_{j=1}^{N} n\nu_j \overline{\zeta\left(z - c_0^{(j)}\right)} + 2\pi n\varepsilon, \tag{3.9.1b}$$

respectively, where $\varepsilon \in \mathbf{C}$ is the complex number corresponding to $\varepsilon \in \mathbf{R}^2$. We now assume that an eigenfunction $f \in L^2_{\omega,n}(\mathbf{R}^2)$ associated with an eigenvalue $E$ can be expressed as the product of holomorphic and anti-holomorphic functions,

$$f(\boldsymbol{x}) = f_1(z) f_2(\overline{z}), \tag{3.9.2}$$

where $f(\boldsymbol{x})$ might be out of the domain $D(H)$. Through the separation of variables, the eigenvalue equation $Hf = Ef$ is broken up into the following ordinary differential equations:

$$\frac{\mathrm{d}f_1(z)}{\mathrm{d}z} = \left\{ \mathrm{i}\left(\sqrt{\frac{E}{2}}\lambda - \pi n\overline{\varepsilon}\right) - \sum_{j=1}^{N} \frac{n\nu_j}{2}\zeta\left(z - c_0^{(j)}\right) \right\} f_1(z), \tag{3.9.3a}$$

$$\frac{\mathrm{d}f_2(\overline{z})}{\mathrm{d}\overline{z}} = \left\{ \mathrm{i}\left(\sqrt{\frac{E}{2}}\lambda^{-1} - \pi n\varepsilon\right) + \sum_{j=1}^{N} \frac{n\nu_j}{2}\overline{\zeta\left(z - c_0^{(j)}\right)} \right\} f_2(\overline{z}), \tag{3.9.3b}$$

where $\lambda \in \mathbf{C}$ is a separation constant. The coefficient functions on the right-hand sides of (3.9.3) have poles of order 1 at $\Lambda$, and its residues are not always integers, so that $f_1(z)$ and $f_2(\overline{z})$ may be multi-valued functions with branch points at $\Lambda$. Solutions to (3.9.3) are given, up to constant factors, by

$$f_1(z) = \exp\left(\sqrt{\frac{E}{2}}\lambda - \pi n\overline{\varepsilon}\right)\mathrm{i}z \prod_{j=1}^{N} \left(\sigma\left(z - c_0^{(j)}\right)\right)^{-\frac{n\nu_j}{2}}, \tag{3.9.4a}$$

$$f_2(\overline{z}) = \exp\left(\sqrt{\frac{E}{2}}\lambda^{-1} - \pi n\varepsilon\right)\mathrm{i}\overline{z} \prod_{j=1}^{N} \overline{\left(\sigma\left(z - c_0^{(j)}\right)\right)^{\frac{n\nu_j}{2}}}, \tag{3.9.4b}$$

where $\sigma(z)$ is the Weierstrass $\sigma$-function [26] defined to be

$$\sigma(z) = z \prod_{(m_1,m_2)\in\mathbf{Z}^2\setminus\{0\}} \left(1 - \frac{z}{m_1 + \mathrm{i}m_2}\right) \exp\left(\frac{z}{m_1 + \mathrm{i}m_2} + \frac{z^2}{2(m_1 + \mathrm{i}m_2)^2}\right). \tag{3.9.5}$$

**Lemma 3.19.** *The Weierstrass $\sigma$-function has the following properties.*

*(1) $\sigma(z)$ is a holomorphic function on $\mathbf{C}$ and has zeros of order 1 at $\mathbf{Z} + \mathrm{i}\mathbf{Z}$.*
*(2) $\sigma(z)$ satisfies*

$$\frac{\mathrm{d}\sigma(z)}{\mathrm{d}z} = \zeta(z)\sigma(z). \tag{3.9.6}$$

*(3) For $m = m_1 + \mathrm{i}m_2 \in \mathbf{Z} + \mathrm{i}\mathbf{Z}$, $\sigma(z)$ has the shift property*

$$\sigma(z + m) = (-1)^{m_1+m_2+m_1m_2} \exp\left(\pi\overline{m}\left(z + \frac{m}{2}\right)\right)\sigma(z). \tag{3.9.7}$$

Though $f_1(z)$ and $f_2(\overline{z})$ given in (3.9.4) are multi-valued, the product $f(\boldsymbol{x}) = f_1(z)f_2(\overline{z})$ is single-valued, if and only if $n\nu_j$, $j = 1, \ldots, N$, are all integers. Further, when $f(\boldsymbol{x})$ is single-valued, the eigenvalue $E$ and the complex parameter $\lambda$ are determined so that $f$ may satisfy the shift property, $f(\boldsymbol{x}+\boldsymbol{m}) = \mathrm{e}^{2\pi\mathrm{i}n\langle \boldsymbol{m},\omega\boldsymbol{x}\rangle} f(\boldsymbol{x})$. The shift property for $f$ together with (3.9.7) results in

$$E_{l_1,l_2} = 2\pi^2\left\{\left(l_1 + n\varepsilon_1 - \frac{nC_2}{2}\right)^2 + \left(l_2 + n\varepsilon_2 + \frac{nC_1}{2}\right)^2\right\}, \tag{3.9.8}$$

$$\lambda = \frac{2\pi}{\sqrt{2E}}\left\{\left(l_1 + n\varepsilon_1 - \frac{nC_2}{2}\right) - \mathrm{i}\left(l_2 + n\varepsilon_2 + \frac{nC_1}{2}\right)\right\}, \tag{3.9.9}$$

where $l_k \in \mathbf{Z}$, $k = 1, 2$, and $C_1 + \mathrm{i}C_2 = \sum_{j=1}^{N} \nu_j c_0^{(j)}$. Then, the eigenfunction associated with (3.9.8) is put in the form

$$f_{l_1,l_2}(\boldsymbol{x}) = \exp\left(2\pi\mathrm{i}\left\{\left(l_1 - \frac{nC_2}{2}\right)x_1 + \left(l_2 + \frac{nC_1}{2}\right)x_2\right\}\right)\prod_{j=1}^{N}\left(\frac{\overline{\sigma\left(x_1 + \mathrm{i}x_2 - c_0^{(j)}\right)}}{\sigma\left(x_1 + \mathrm{i}x_2 - c_0^{(j)}\right)}\right)^{\frac{n\nu_j}{2}}. \tag{3.9.10}$$

While we have obtained explicitly eigenvalues and eigenfunctions for the A-B Hamiltonian with all fluxes quantized, the deficiency subspaces $\mathcal{K}_\pm$ and domain of the closure $\overline{H}$ are unidentified yet. At this stage, we have the following.

**Theorem 3.20.** *Let $n\nu_j \in \mathbf{Z}$, $j = 1, \ldots, N$. Then, the adjoint operator $H^*$ to the A-B Hamiltonian (3.8.2) associated with the A-B potential (3.8.1) has eigenvalues and the associated eigenfunctions given by (3.9.8) and (3.9.10), respectively.*

We here make a comment on the A-B operator with all fluxes quantized. In view of the eigenfunctions (3.9.10), we can construct a unitary operator $U$ which brings $H$ into a formally self-adjoint operator. In fact, we define a multiplication operator $U$ of $L^2_{\omega,n}(\mathbf{R}^2)$ to $L^2_{0,n}(\mathbf{R}^2)$ to be

$$U = \mathrm{e}^{-\pi\mathrm{i}n(C_2x_1 - C_1x_2)}\prod_{j=1}^{N}\left(\frac{\sigma\left(x_1 + \mathrm{i}x_2 - c_0^{(j)}\right)}{\overline{\sigma\left(x_1 + \mathrm{i}x_2 - c_0^{(j)}\right)}}\right)^{\frac{n\nu_j}{2}}, \tag{3.9.11}$$

where $n\nu_j$, $j = 1, 2, \ldots, N$, are all integers. Then $U$ brings the Hamiltonian $H$ into the Hamiltonian $H'$ on $L^2_{0,n}(\mathbf{R}^2)$,

$$H' = \frac{1}{2}\left\{\left(-\mathrm{i}\frac{\partial}{\partial x_1} + 2\pi n\varepsilon_1 - \pi nC_2\right)^2 + \left(-\mathrm{i}\frac{\partial}{\partial x_2} + 2\pi n\varepsilon_2 + \pi nC_1\right)^2\right\}, \tag{3.9.12}$$

which is formally self-adjoint. Thus, the Schrödinger equation $Hf = Ef$, $f \in L^2_{\omega,n}(\mathbf{R}^2)$, is brought into the equation $H'g = Eg$, $g \in L^2_{0,n}(\mathbf{R}^2)$. According to the theory of Fourier analysis, the Hilbert space

$$L^2_{0,n}(\mathbf{R}^2) = \left\{g : \mathbf{R}^2 \to \mathbf{C} \mid g(\boldsymbol{x}+\boldsymbol{m}) = g(\boldsymbol{x}), \boldsymbol{m} \in \mathbf{Z}^2, \int_{I^2} |g(\boldsymbol{x})|^2\,\mathrm{d}^2\boldsymbol{x} < +\infty\right\} \tag{3.9.13}$$

has a complete orthonormal system $\{\mathrm{e}^{2\pi\mathrm{i}\langle \boldsymbol{l},\boldsymbol{x}\rangle}\}_{\boldsymbol{l}\in\mathbf{Z}^2}$, which gives eigenfunctions of $H'$ with the eigenvalues (3.9.8). This observation implies that the eigenfunctions (3.9.10) also form a complete orthonormal system in $L^2_{\omega,n}(\mathbf{R}^2)$.

However, we have here to remark that the operator $H'$ is not self-adjoint, since theorem 3.18 shows that $H'$ should have the deficiency indices $(N, N)$ with $M = 0$. This is because the domain $D(H')$ is $\left\{g \in C^{\infty}_{0,n}(\dot{\mathbf{R}}^2) \,\middle|\, \operatorname{supp} g \cap \Lambda \neq \emptyset\right\}$. We add further a remark on $H'$. Let

$$\alpha_1 = \varepsilon_1 - \frac{C_2}{2}, \qquad \alpha_2 = \varepsilon_2 + \frac{C_1}{2}. \tag{3.9.14}$$

The operator $H'$ is associated with the constant connection $A_0 = \alpha_1\,\mathrm{d}x_1 + \alpha_2\,\mathrm{d}x_2$ which is in $\mathcal{A}_0(\dot{\mathbf{R}}^2)$. According to theorem 3.14, the quantum system with the Hamiltonian $H'$ is unitarily equivalent to a free particle system on the punctured 2-torus, if and only if $\mathrm{e}^{2n\pi\mathrm{i}\alpha_k} = 1, k = 1, 2$. Hence, $H'$ is not unitarily equivalent to the free particle Hamiltonian, if $n\alpha_k$ are not integers.

### *3.10. Concluding remarks*

We have generalized the A-B potential on the plane $\mathbf{R}^2\backslash\{0\}$ to flat connections on circle bundles over a punctured 2-torus and have classified those connections. On the basis of the connection theory, we have set up quantum systems on the associated complex line bundles to study the A-B Hamiltonian operator. The crucial role in the geometry and analysis of the generalized A-B system is played by the fluxes of solenoids, where the flux is defined as the integral of the connection form along a small curve around the singular point.

In proposition 2.5, we have shown that the sum of fluxes is quantized to be the integer that is characteristic of the circle bundle over the punctured 2-torus. The fluxes and the integral of the flat connection along the cycles of the 2-torus are put together to identify the moduli space of the flat connections with $T^{N+1}$, which is shown in theorem 2.10, where $N$ is the number of solenoids. In particular, we have given, in theorem 2.11, the formula that describes explicitly an arbitrary flat connection.

Turning to quantum systems, we have shown that all quantum systems which are defined on associated complex line bundles over the punctured 2-torus are characterized by the 'representations' of the moduli space $T^{N+1} \cong U(1)^{N+1}$, which is described in theorem 3.14 or in theorem 3.15. The A-B Hamiltonian is defined in association with the flat connection. We have shown in theorem 3.18 that the deficiency indices of the A-B Hamiltonian are $(M + N, M + N)$, where $M$ is the number of the non-quantized fluxes and where the flux $\nu_j$ is called quantized, if $n\nu_j$ is an integer, $n$ being the integer characteristic of the associated line bundle. To show this theorem, we have used the localization lemma together with the well-known fact on the deficiency indices for the usual A-B Hamiltonian.

For the adjoint operator to the A-B Hamiltonian with all fluxes quantized, we have obtained eigenvalues and associated eigenfunctions in terms of the Weierstrass sigma functions. However, the A-B Hamiltonian with all fluxes quantized has deficiency indices $(N, N)$, so that it might have another eigenvalue. We have not studied the deficiency subspaces. Let us be reminded that the Hamiltonian in question is unitarily transformed into $H'$, given by (3.9.12). Though $H'$ is not unitarily equivalent, in general, to the free particle Hamiltonian on the punctured 2-torus, results on the finitely many point interactions in two dimensions [4] would be of help in studying the deficiency subspaces in our case. However, it is reserved for future study.

It remains open to find solutions to the Schrödinger equation with non-quantized fluxes. To determine self-adjoint extensions is hard to study even for the usual A-B Hamiltonian [7, 9]. To address this problem, it would be useful to take a Riemann surface covering the punctured 2-torus in order to describe eigenfunctions. The covering space method was taken in [21] for the A-B Hamiltonian on the punctured plane. This method was developed further for the Pauli operator for the A-B effect with two solenoids [12].

We have studied the A-B Hamiltonian on the punctured 2-torus, but it is possible to generalize the Hamiltonian in association with the A-B connection on the puncture torus plus the connection whose curvature is a constant magnetic field. The Hamiltonian of this type on a puncture plane has been already studied in [10, 14, 16], and the Hamiltonian for a constant magnetic field on the $n$-torus was analysed in [22, 23].

### A.1. Localization lemma

In this appendix, we prove lemma 3.17, which asserts that the total deficiency indices are determined by localizing the domain of the A-B Hamiltonian in the vicinity of singularities.

*Appendix .1. Notation*

In this appendix, for simplicity and for convenience, we use the following notation:

$$\boldsymbol{A} = (A_1, A_2)^{\mathrm{T}}, \tag{A.1.1}$$

$$\boldsymbol{P} = (P_1, P_2)^{\mathrm{T}}, \tag{A.1.2}$$

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)^{\mathrm{T}}, \tag{A.1.3}$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \tag{A.1.4}$$

$$\mathrm{supp}(\boldsymbol{\nabla} f) = \mathrm{supp}\left(\frac{\partial f}{\partial x_1}\right) \cup \mathrm{supp}\left(\frac{\partial f}{\partial x_2}\right), \tag{A.1.5}$$

$$\langle \nabla f(\boldsymbol{x}), \nabla g(\boldsymbol{x})\rangle = \sum_{k=1}^{2} \frac{\partial f}{\partial x_k}(\boldsymbol{x}) \frac{\partial g}{\partial x_k}(\boldsymbol{x}), \tag{A.1.6}$$

where the superscript 'T' indicates the transpose.

*A.2. Surjectivity of $\iota : \mathcal{D}(H_O^*)/\mathcal{D}(\overline{H_O}) \longrightarrow D(H^*)/D(\overline{H})$*

Let $\mu_j$, $j = 1, \ldots, N$, be smooth functions on $\mathbf{R}^2$ which satisfy that

$$\mu_j(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \left|\boldsymbol{x} - \boldsymbol{c}_{\boldsymbol{0}}^{(j)}\right| < \dfrac{\epsilon}{2}, \\ 0, & \text{if } \left|\boldsymbol{x} - \boldsymbol{c}_{\boldsymbol{0}}^{(j)}\right| \geqslant \dfrac{2\epsilon}{3} \end{cases} \tag{A.2.1}$$

and that $0 \leqslant \mu_j(\boldsymbol{x}) \leqslant 1$ for $\boldsymbol{x} \in \mathbf{R}^2$. Putting $\tilde{\mu}_j(\boldsymbol{x}) = \sum_{\boldsymbol{m}\in\mathbf{Z}^2} \mu_j(\boldsymbol{x} - \boldsymbol{m})$, we define a cut-off function $\tilde{\mu}$ by

$$\tilde{\mu} = \sum_{j=1}^{N} \tilde{\mu}_j. \tag{A.2.2}$$

The proof of the surjectivity is outlined as follows: For an arbitrary $f \in D(H^*)$, the support of $\tilde{\mu} f$ is a compact subset of $O$. If $\tilde{\mu} f \in \mathcal{D}(H_O^*)$ and if $(1 - \tilde{\mu}) f \in D(\overline{H})$, then $\iota$ maps the equivalence class $\tilde{\mu} f + \mathcal{D}(\overline{H_O})$ to $\tilde{\mu} f + D(\overline{H}) = \tilde{\mu} f + (1 - \tilde{\mu}) f + D(\overline{H}) = f + D(\overline{H})$. Thus, the linear map $\iota$ will prove to be surjective.

To begin with, we show that $\tilde{\mu} f \in \mathcal{D}(H_O^*)$ for any $f \in D(H^*)$. We first note that the A-B Hamiltonian is locally gauge transformed into minus half the standard Laplacian, $-\frac{1}{2}\Delta$, outside the singularity. In fact, since the set $\mathrm{supp}(\boldsymbol{\nabla}\tilde{\mu}) \cap [0, 1]^2$ can be covered by a finite number of contractible bounded open sets $\{U_\lambda\}$ in $\dot{\mathbf{R}}^2$, and since the closed 1-form $A$ becomes exact on a contractible open set $U_\lambda$, there exists a smooth function $h_\lambda$ such that $A_k = \partial h_\lambda/\partial x_k$, $k = 1, 2$, on $U_\lambda$, which gives rise to a local gauge transformation $f \mapsto \mathrm{e}^{-2\pi \mathrm{i} n h_\lambda} f$ on $U_\lambda$ such that

$$\boldsymbol{P} = \mathrm{e}^{-2\pi \mathrm{i} n h_\lambda}(-\mathrm{i}\nabla)\,\mathrm{e}^{2\pi \mathrm{i} n h_\lambda}. \tag{A.2.3}$$

It then follows that $H = -\frac{1}{2}\,\mathrm{e}^{-2\pi \mathrm{i} n h_\lambda}\Delta\,\mathrm{e}^{2\pi \mathrm{i} n h_\lambda}$ on $C_0^\infty(U_\lambda)$. Hence, for $f \in D(H^*)$ and for any $\xi \in C_0^\infty(U_\lambda)$, we have

$$\begin{aligned}\int_{U_\lambda} \overline{(H^* f)(\boldsymbol{x})}\,\mathrm{e}^{-2\pi \mathrm{i} n h_\lambda(\boldsymbol{x})}\xi(\boldsymbol{x})\,\mathrm{d}^2\boldsymbol{x} &= \int_{U_\lambda} \overline{f(\boldsymbol{x})} H(\mathrm{e}^{-2\pi \mathrm{i} n h_\lambda}\xi)(\boldsymbol{x})\,\mathrm{d}^2\boldsymbol{x} \\ &= -\frac{1}{2}\int_{U_\lambda} \overline{f(\boldsymbol{x})}\,\mathrm{e}^{-2\pi \mathrm{i} n h_\lambda(\boldsymbol{x})}(\Delta\xi)(\boldsymbol{x})\,\mathrm{d}^2\boldsymbol{x}.\end{aligned} \tag{A.2.4}$$

This implies that $\phi_\lambda =: \mathrm{e}^{2\pi \mathrm{i} n h_\lambda} f|_{U_\lambda}$ satisfies the equation

$$-\tfrac{1}{2}\Delta\phi_\lambda = \mathrm{e}^{2\pi \mathrm{i} n h_\lambda}(H^* f)|_{U_\lambda} \in L^2(U_\lambda), \tag{A.2.5}$$

in the weak sense. According to a theorem on Sobolev spaces [19], this equation implies that $\phi_\lambda \in W^2(U_\lambda)$, where $W^2(U_\lambda)$ denotes the Sobolev space of order 2 on $U_\lambda$ [18, 19].

We proceed to the function $\tilde{\mu} f$ and a functional determined thereby. For any $g \in D(H)$, we have

$$\begin{aligned}&\left|\langle \tilde{\mu} f, Hg\rangle_{L^2_{\omega,n}(\mathbf{R}^2)}\right| \\ &\quad= \left|\int_{I^2\cap \operatorname{supp}\tilde{\mu}} \overline{\tilde{\mu}(\boldsymbol{x}) f(\boldsymbol{x})}(Hg)(\boldsymbol{x})\,\mathrm{d}^2\boldsymbol{x}\right| \\ &\quad= \left|\int_{I^2\cap \operatorname{supp}\tilde{\mu}} \overline{f(\boldsymbol{x})}\left(H(\tilde{\mu} g) - \langle -\mathrm{i}\nabla\tilde{\mu}, \boldsymbol{P} g\rangle + \frac{1}{2} g\Delta\tilde{\mu}\right)(\boldsymbol{x})\,\mathrm{d}^2\boldsymbol{x}\right| \\ &\quad\leqslant \left|\langle \tilde{\mu} H^* f + \frac{1}{2} f\Delta\tilde{\mu}, g\rangle_{L^2_{\omega,n}(\mathbf{R}^2)}\right| + \left|\int_{I^2\cap \operatorname{supp}\tilde{\mu}} \overline{f(\boldsymbol{x})}\langle \mathrm{i}\nabla\tilde{\mu}(\boldsymbol{x}), (\boldsymbol{P} g)(\boldsymbol{x})\rangle\,\mathrm{d}^2\boldsymbol{x}\right| \\ &\quad\leqslant \left\|\tilde{\mu} H^* f + \frac{1}{2} f\Delta\tilde{\mu}\right\|_{L^2_{\omega,n}(\mathbf{R}^2)} \|g\|_{L^2_{\omega,n}(\mathbf{R}^2)} + \left|\int_{I^2\cap \operatorname{supp}\tilde{\mu}} \overline{f(\boldsymbol{x})}\langle \mathrm{i}\nabla\tilde{\mu}(\boldsymbol{x}), (\boldsymbol{P} g)(\boldsymbol{x})\rangle\,\mathrm{d}^2\boldsymbol{x}\right|.\end{aligned} \tag{A.2.6}$$

We now work with the integral in the second term on the right-hand side, restricting $I^2\cap\operatorname{supp}\tilde{\mu}$ to $U_\lambda \cap \operatorname{supp}\tilde{\mu}$:

$$\begin{aligned}&\int_{U_\lambda\cap \operatorname{supp}\tilde{\mu}} \overline{f(\boldsymbol{x})}\langle \mathrm{i}\nabla\tilde{\mu}(\boldsymbol{x}), (\boldsymbol{P} g)(\boldsymbol{x})\rangle\,\mathrm{d}^2\boldsymbol{x} \\ &\quad= \int_{U_\lambda\cap \operatorname{supp}\tilde{\mu}} \overline{\mathrm{e}^{2\pi \mathrm{i} n h_\lambda(\boldsymbol{x})} f(\boldsymbol{x})}\langle \mathrm{i}\nabla\tilde{\mu}(\boldsymbol{x}), -\mathrm{i}\nabla(\mathrm{e}^{2\pi \mathrm{i} n h_\lambda} g)(\boldsymbol{x})\rangle\,\mathrm{d}^2\boldsymbol{x} \\ &\quad= \int_{U_\lambda\cap \operatorname{supp}\tilde{\mu}} \sum_{k=1}^{2} \overline{\phi_\lambda(\boldsymbol{x})}\frac{\partial\tilde{\mu}}{\partial x_k}\frac{\partial}{\partial x_k}(\mathrm{e}^{2\pi \mathrm{i} n h_\lambda(\boldsymbol{x})} g(\boldsymbol{x}))\,\mathrm{d}^2\boldsymbol{x} \\ &\quad= -\int_{U_\lambda\cap \operatorname{supp}\tilde{\mu}} \sum_{k=1}^{2} \overline{\frac{\partial}{\partial x_k}\left(\phi_\lambda\frac{\partial\tilde{\mu}}{\partial x_k}\right)(\boldsymbol{x})}\,\mathrm{e}^{2\pi \mathrm{i} n h_\lambda(\boldsymbol{x})} g(\boldsymbol{x})\,\mathrm{d}^2\boldsymbol{x} \\ &\quad= \int_{U_\lambda\cap \operatorname{supp}\tilde{\mu}} (\overline{\mathrm{e}^{-2\pi \mathrm{i} n h_\lambda(\boldsymbol{x})}\langle -\mathrm{i}\nabla\phi_\lambda(\boldsymbol{x}), -\mathrm{i}\nabla\tilde{\mu}(\boldsymbol{x})\rangle} g(\boldsymbol{x}) - \overline{(f\Delta\tilde{\mu})(\boldsymbol{x})} g(\boldsymbol{x}))\,\mathrm{d}^2\boldsymbol{x}.\end{aligned} \tag{A.2.7}$$

Hence, we obtain an estimate of the second term on the right-hand side of (A.2.6),

$$\left|\int_{I^2\cap\operatorname{supp}\tilde{\mu}} \overline{f(\boldsymbol{x})}\langle \mathrm{i}\nabla\tilde{\mu}(\boldsymbol{x}), (\boldsymbol{P}g)(\boldsymbol{x})\rangle\, \mathrm{d}^2\boldsymbol{x}\right|$$
$$\leqslant \sum_{\lambda}^{\text{finite}} \int_{U_\lambda\cap\operatorname{supp}\tilde{\mu}} (|\langle\nabla\phi_\lambda, \nabla\tilde{\mu}\rangle||g(\boldsymbol{x})| + |f\Delta\tilde{\mu}||g(\boldsymbol{x})|)\, \mathrm{d}^2\boldsymbol{x}$$
$$\leqslant \sum_{\lambda}^{\text{finite}} \left(\left(\int_{U_\lambda\cap\operatorname{supp}\tilde{\mu}} |\langle\nabla\phi_\lambda, \nabla\tilde{\mu}\rangle|^2\, \mathrm{d}^2\boldsymbol{x}\right)^{1/2} + \left(\int_{U_\lambda\cap\operatorname{supp}\tilde{\mu}} |f\Delta\tilde{\mu}|^2\, \mathrm{d}^2\boldsymbol{x}\right)^{1/2}\right)\|g\|_{L^2_{\omega,n}(\mathbf{R}^2)}. \tag{A.2.8}$$

The inequalities (A.2.6) and (A.2.8) are put together to imply that the linear functional $D(H) \to \mathbf{C} : g \mapsto \langle\tilde{\mu}f, Hg\rangle_{L^2_{\omega,n}(\mathbf{R}^2)}$ is bounded. According to Riesz's representation theorem, there is a unique element $f' \in L^2_{\omega,n}(\mathbf{R}^2)$ such that $\langle\tilde{\mu}f, Hg\rangle_{L^2_{\omega,n}(\mathbf{R}^2)} = \langle f', g\rangle_{L^2_{\omega,n}(\mathbf{R}^2)}$ for any $g \in D(H)$, that is, $f' = H^*(\tilde{\mu}f)$. Then, $\tilde{\mu}f$ is in $D(H^*)$.

If we start with $g \in D(H_O)$ in the functional $g \mapsto \langle\tilde{\mu}f, Hg\rangle_{L^2_{\omega,n}(\mathbf{R}^2)}$, we will be led to the conclusion that $\tilde{\mu}f \in \mathcal{D}(H_O^*)$ in the same manner.

Our next task is to prove that $(1-\tilde{\mu})f \in D(\overline{H})$ for $f \in D(H^*)$. For some open set $O'$ in $\mathbf{R}^2$ with $\Lambda \subset O' \subset \mathbf{R}^2\backslash\operatorname{supp}(1-\tilde{\mu})$, we choose a non-singular 1-form $A' \in \mathcal{A}_\omega(\mathbf{R}^2)$ which satisfies

$$A'|_{\mathbf{R}^2\backslash O'} = A|_{\mathbf{R}^2\backslash O'}. \tag{A.2.9}$$

The Hamiltonian $H'$ associated with $A'$ is given by

$$H' = \frac{1}{2}\sum_{k=1,2}\left(-\mathrm{i}\frac{\partial}{\partial x_k} + 2\pi n A'_k\right)^2 \qquad \text{with} \qquad D(H') = C^\infty_{\omega,n}(\mathbf{R}^2), \tag{A.2.10}$$

which is known to be essentially self-adjoint [19, 20]. Since $(1-\tilde{\mu})f \in D(H^*)$ and since $H^*(1-\tilde{\mu})f = (H')^*(1-\tilde{\mu})f = \overline{H'}(1-\tilde{\mu})f$, there is a sequence $\{\eta_m\} \subset C^\infty_{\omega,m}(\mathbf{R}^2)$ such that

$$\eta_n \to (1-\tilde{\mu})f \qquad \text{and} \qquad H'\eta_m \to H^*(1-\tilde{\mu})f, \qquad \text{as} \quad m \to \infty. \tag{A.2.11}$$

Let $q \in C^\infty(\mathbf{R}^2)$ be a periodic function which is equal to 1 on a neighbourhood of $\operatorname{supp}(1-\tilde{\mu})$ and whose support is contained in $\mathbf{R}^2\backslash O'$. Let $\xi_m = q\eta_m$. Then $\xi_m \in D(H)$, as is easily seen. We verify that $\xi_m$ as well as $\eta_m$ converges to $(1-\tilde{\mu})f$. In fact, we have

$$\begin{aligned}\|\xi_m - (1-\tilde{\mu})f\|_{L^2_{\omega,n}(\mathbf{R}^2)} &= \|q\eta_m - q(1-\tilde{\mu})f\|_{L^2_{\omega,n}(\mathbf{R}^2)}\\ &\leqslant C\|\eta_m - (1-\tilde{\mu})f\|_{L^2_{\omega,n}(\mathbf{R}^2)} \to 0, \qquad \text{as } m \to \infty,\end{aligned} \tag{A.2.12}$$

where $C$ is a positive constant. We now show that the sequence $H\xi_m$ converges. To this end, we note that $H\xi_m$ is put in the form

$$\begin{aligned}H\xi_m &= H(q\eta_m) = H'(q\eta_m)\\ &= q(H'\eta_m) + \langle -\mathrm{i}\nabla q, \boldsymbol{P}'\eta_m\rangle - \tfrac{1}{2}\eta_m\Delta q,\end{aligned} \tag{A.2.13}$$

where

$$\boldsymbol{P}'\eta_m = -\mathrm{i}\nabla\eta_m + 2\pi n\eta_m\boldsymbol{A}'. \tag{A.2.14}$$

We now deal with the second term on the right-hand side of (A.19). The squared norm of this term is estimated as

$$\begin{aligned}\int_{I^2} |\langle -\mathrm{i}\nabla q, \boldsymbol{P}'\eta_m\rangle|^2\,\mathrm{d}^2\boldsymbol{x} &\leqslant \int_{I^2} |\nabla q|^2 |\boldsymbol{P}'\eta_m|^2\,\mathrm{d}^2\boldsymbol{x} \\ &\leqslant C\int_{I^2} |\boldsymbol{P}'\eta_m|^2\,\mathrm{d}^2\boldsymbol{x} \\ &= 2C\int_{I^2} \overline{\eta_m} H'\eta_m\,\mathrm{d}^2\boldsymbol{x}, \end{aligned} \tag{A.2.15}$$

where $C$ is a positive constant and we have used the fact that $|\nabla q|$ is bounded, and where the last equality holds since integration by part is well applied on account of the shift property of $A' \in \mathcal{A}_\omega(\mathbf{R}^2)$ and of $\eta_m \in C^\infty_{\omega,n}(\mathbf{R}^2)$. From (A.19) and (A.21), it follows that

$$\begin{aligned}\|H\xi_k - H\xi_m\|_{L^2_{\omega,n}(\mathbf{R}^2)} &\leqslant \|qH'(\eta_k - \eta_m)\|_{L^2_{\omega,n}(\mathbf{R}^2)} + \|\langle \nabla q, \boldsymbol{P}'(\eta_k - \eta_m)\rangle\| \\ &\quad + \tfrac{1}{2}\|(\eta_k - \eta_m)\Delta q\|_{L^2_{\omega,n}(\mathbf{R}^2)} \leqslant C_1\|H'(\eta_k - \eta_m)\|_{L^2_{\omega,n}(\mathbf{R}^2)} \\ &\quad + C_2\langle \eta_n - \eta_m, H'(\eta_n - \eta_m)\rangle^{1/2}_{L^2_{\omega,n}(\mathbf{R}^2)} + C_3\|\eta_k - \eta_m\|_{L^2_{\omega,n}(\mathbf{R}^2)}, \end{aligned} \tag{A.2.16}$$

where $C_1$, $C_2$ and $C_3$ are positive constants. This shows that the sequence $\{H\xi_m\}$ converges on account of (A.17). It then turns out that $\lim_{m\to\infty}\xi_m = (1 - \tilde{\mu})f$ is in the domain of $\overline{H}$.

*A.3. Injectivity of* $\iota : \mathcal{D}(H_O^*)/\mathcal{D}(\overline{H_O}) \longrightarrow D(H^*)/D(\overline{H})$

The injectivity of the map $\iota$ is equivalent to $\ker\iota = \{0\}$, i.e.,

$$f \in \mathcal{D}(H_O^*) \cap D(\overline{H}) \quad\Longrightarrow\quad f \in \mathcal{D}(\overline{H_O}). \tag{A.3.1}$$

We assume that $f$ is in $\mathcal{D}(H_O^*) \cap D(\overline{H})$. Since $f \in D(\overline{H})$, there exists a sequence $\{\eta_m\} \subset D(H)$ such that

$$\eta_m \to f \qquad \text{and} \qquad H\eta_m \to \overline{H}f, \qquad \text{as } m \to \infty. \tag{A.3.2}$$

We pick a periodic function $q \in C^\infty(\mathbf{R}^2)$ satisfying the following conditions:

(i) $0 \leqslant q(\boldsymbol{x}) \leqslant 1$ for any $\boldsymbol{x} \in \mathbf{R}^2$,
(ii) $q$ is equal to 1 on a neighbourhood of $\operatorname{supp} f$,
(iii) $\operatorname{supp} f \subsetneq \operatorname{supp} q \subset O$, and
(iv) the interior of $\operatorname{supp} q$ contains $\Lambda$,

where we have used the fact that $\operatorname{supp} f \subset O$ on account of $f \in \mathcal{D}(H_O^*)$. In the same manner as in the proof of the surjectivity, we can show that $\xi_m = q\eta_m \in D(H)$ converges to $f$ and that $H\xi_m$ converges as well. Then $f \in \mathcal{D}(H_O^*) \cap D(\overline{H})$ is also an element of $\mathcal{D}(\overline{H_O})$. The detail of the proof is omitted.

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